Department of Mathematics, University of Houston
Math 6397. Algebraic Topology. David Blecher
Homework 1

1. If X is path connected, and $\hat{\beta}$ is as defined in 3.2.8, show that $\Pi(X, x_0)$ is abelian iff $\hat{\alpha} = \hat{\beta}$ for every pair of paths α, β from x_0 to another point $x_1 \in X$.

2. Let $h : X \to Y$ be continuous, with $h(x_0) = y_0$ and $h(x_1) = y_1$. Let α be a path in X from x_0 to x_1, and let $\beta = h \circ \alpha$. Show that $\widehat{\beta} \circ (h_{x_0})_* = (h_{x_1})_* \circ \widehat{\alpha}$. Here we are writing $(h_x)_*$ for h_* as a map on $\pi_1(X, x)$.

3. Let X be path connected and $a, b, c \in X$. Show that any path in X from a to b is path homotopic to a path in X from a to b that passes through c.

4. Let $p : E \to B$ be continuous and surjective. Suppose that U is an open set of B that is evenly covered by p. Show that if U is connected, then the partition of $p^{-1}(U)$ into slices is unique.

5. Let $p : E \to B$ be a covering map ; let B be connected. Show that if $p^{-1}(\{b_0\})$ has k elements for some $b_0 \in B$, then $p^{-1}(\{b\})$ has k elements for every $b \in B$.

6. Let $p : E \to B$ be a covering map with E path connected. Show that if B is simply connected, then p is a homeomorphism.

7. In class we briefly discussed the annulus, $\bar{B}((0,0),2) \setminus B((0,0),1)$, a closed disk with a concentric open disk removed. Show that a covering space for this is a closed infinite strip $R \times [1,2]$, and using this and the lifting correspondence theorem show that the fundamental group of the annulus is the integers.

8. Let $p : E \to B$ be a covering map. Show: (a) If B is Hausdorff, regular, completely regular, or locally compact Hausdorff, then so is E. (b) If B is compact and $p^{-1}(\{b\})$ is finite for each $b \in B$, then E is compact.

9. Show that any path from a to b in an open set U in R^n is path homotopic in U to a path consisting of straight line segments (a 'polygonal path').

10. Show that a product of two simply connected spaces is simply connected. Show that being simply connected is a topological but not hereditary property.

11. Show that $R^n \setminus \{0\}$ is simply connected if $n \geq 3$. What is $\pi_1(R^2 \setminus \{0\})$? Prove it.

12. Show that R^1 and R^n are not homeomorphic if $n > 1$. Show that R^2 and R^n are not homeomorphic if $n > 2$.

13. What is wrong with the following approach to proving S^2 is simply connected: If γ is a loop on S^2, and p is a point not on this loop, then by stereographic projection $S^2 \setminus \{p\} \cong R^2$. Now R^2 is convex, so γ is path homotopic to a constant loop by 3.2.3.