Department of Mathematics, University of Houston
Math 6397. Algebraic Topology. David Blecher
Homework 2

1. Prove the $n = 1$ case of the Borsuk-Ulam theorem: For continuous map $f : S^1 \to R$ there exists $x \in S^1$ with $f(x) = f(-x)$.

2. Let F be a continuous function from S^2 to R^3. Show using the Hairy ball theorem that there is a point $p \in S^2$ at which $F(p)$ is normal to S^2, that is, $F(p)$ is a multiple of $\vec{p}$.

3. Show that contractible spaces are simply connected.

4. Prove that if Y is contractible, then any two maps from X to Y are homotopic. Prove that if X is contractible and Y is path-connected, then any two maps from X to Y are homotopic.

5. If X is path-connected and $\pi_1(X)$ is commutative, show that there is a one-to-one correspondence between elements of $\pi_1(X)$ and homotopy (as opposed to path homotopy) classes of loops in X.

6. Let $g : S^2 \to S^2$ be continuous. If $g(x) \neq g(-x)$ for all $x \in S^2$, show that g is surjective.

7. Prove that a space is path connected if it has a deformation retract that is path connected.

8. Prove that the doubly punctured plane(R^2 with two points removed) is homotopically equivalent to the figure eight. Show that its fundamental group is the free group on 2 generators.

9. Show that the fundamental group of the double torus is not abelian.

10. Show that the winding number is zero for points in the unbounded connected component of a loop γ around a point p. Show that the winding number for points in the bounded component is ± 1 for piecewise smooth Jordan curves, e.g. complete the sketch proof given in class for this that picked a small disk D center a in the bounded component.

11. Show that for a loop f in $\mathbb{C} \setminus \{0\}$ there exists $\epsilon > 0$ such that any loop g with $|g(z) - f(z)| < \epsilon$ for all z, has the same winding number as f. Show that if w is a point not lying on a loop f in $\mathbb{C}$, and if g is another loop satisfying $|g(z) - f(z)| < |w - f(z)|$ for all z then g has the same winding number as f.

12. Show that the bounded component U of the complement of a Jordan curve is the interior of the closure of U. Show that the complement of the components in the Jordan curve theorem are connected. Show that the complement of a compact contractible subset of R^2, is connected (Hint from Caleb 1: Borsuk lemma).

13. Example 4 in Munkes page 359.