Chapter 8. Integration techniques

8.1: Using integral tables.

Many integrals can be done simply by looking at the table of integrals on the course website. For example, $\int \frac{\sqrt{9-x^2}}{x^2}dx = -\frac{\sqrt{9-x^2}}{x} - \arcsin\frac{x}{3} + C$ by formula 90 in those tables. Sometimes of course we have to work with an integral a bit to get it into the form we see in the tables. For example, $\int \frac{\sqrt{9-4x^2}}{x^2}dx$. Let $u = 2x$ then $du = 2dx$, $x = u/2$ and $dx = \frac{1}{2}du$, and the integral becomes

$$\frac{1}{2}\int \frac{\sqrt{9-u^2}}{\frac{u^2}{4}}\,dx = 2\int \frac{\sqrt{9-u^2}}{u^2}du = -2\frac{\sqrt{9-u^2}}{u} - 2\arcsin\frac{u}{3} + C = -\frac{\sqrt{9-4x^2}}{x} - 2\arcsin\frac{2x}{3} + C.$$

8.2: Integration by parts.

The integration by parts formula is

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

To prove this note that $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$. Integrating, we get

$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx,$$

and so $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$.

- Example 1: Find $\int x\cos x\,dx$.

  Solution: $\int x\cos x\,dx = \int x\frac{d}{dx}(\sin x)\,dx = x\sin x - \int 1\cdot\sin x\,dx = x\sin x + \cos x$.

- Usually we write the integration by parts formula differently:

  $$\int u\,dv = uv - \int v\,du.$$

  [To see this is the same, let $u = f(x), v = g(x)$, then $du = f'(x)dx, dv = g'(x)dx$, and the last formula then reads $\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$.]

- Example 2: In this notation, evaluate $\int xe^x\,dx$.

  Solution: Let $u = x, dv = e^x\,dx$. Then $du = 1dx = dx$, and integrating dv gives $v = \int e^x\,dx = e^x$. So

  $$\int xe^x\,dx = \int u\,dv = uv - \int v\,du = xe^x - \int e^x dx = xe^x - e^x + C.$$

- Example 3: Find $\int_0^1 xe^x\,dx$.

  Solution: $[xe^x - e^x]_0^1 = e^1 - e^1 - [0 - e^0] = 1$.

- Example 4: Find $\int x^2\ln x\,dx$.

  Solution: The obvious choice here is $u = x^2$ and $dv = \ln x\,dx$. But this isnt so good because it will force us to integrate $\ln x$. Rather let us write $\int x^2\ln x\,dx = \int(\ln x)x^2\,dx$, and let $u = \ln x, dv = x^2\,dx$. Now $du = \frac{1}{x}dx$ and integrating dv gives $v = \int x^2\,dx = \frac{1}{3}x^3$. So

  $$\int(\ln x)x^2\,dx = \int u\,dv = uv - \int v\,du = (\ln x)\cdot\frac{1}{3}x^3 - \int \frac{1}{3}x^3\cdot\frac{1}{x}dx = \frac{x^3\ln x}{3} - \frac{1}{3}\int x^2dx,$$

  which is $\frac{x^3\ln x}{3} - \frac{1}{9}x^3 + C$.

- Example 5: Find $\int \ln x \, dx$.

  Solution: Let us write $\int \ln x \, dx = \int (\ln x) \cdot 1 \, dx$, and let $u = \ln x$ and $dv = 1 \, dx$. Then $du = \frac{1}{x} dx$ and integrating dv gives $v = x$, and so

$$\int \ln x \, dx = \int u \, dv = uv - \int v \, du = (\ln x) \cdot x - \int x \frac{1}{x} dx = x \ln x - \int 1 dx = x \ln x - x + C.$$

- Example 6: Evaluate $\int_0^1 \arcsin x \, dx$.

  Solution: Similar to the last: let $u = \arcsin x$, and $dv = dx$. Then $du = \frac{1}{\sqrt{1-x^2}} dx$, and integrating dv gives $v = x$, and so

$$\int u \, dv = uv - \int v \, du = x \arcsin x - \int x \frac{1}{\sqrt{1-x^2}} dx.$$

  Let $w = 1 - x^2$ then $dw = -2x dx$ and $2x dx = -\frac{1}{2} dw$. The integral becomes:

$$x \arcsin x + \frac{1}{2} \int \frac{1}{\sqrt{w}} \, dw = x \arcsin x + \frac{1}{2} \int w^{-\frac{1}{2}} \, dw = x \arcsin x + w^{\frac{1}{2}} = x \arcsin x + \sqrt{1-x^2}.$$

  Our final answer is $(x \arcsin x + \sqrt{1-x^2})]_0^1 = 1 \arcsin 1 - 0 - \sqrt{1} = \frac{\pi}{2} - 1$.

- Example 7: Evaluate $\int_0^1 x^2 e^x \, dx$.

  Solution: Let $u = x^2, dv = e^x \, dx$ then $du = 2x dx, v = e^x$, and our integral becomes

$$uv - \int v \, du = x^2 e^x - 2 \int e^x x dx.$$

  We did $\int x e^x dx$ earlier, and so our integral becomes

$$(x^2 e^x - 2(x e^x - e^x))]_0^1 = e - 2(e - e) - (-2(-1)) = e - 2.$$

- Example 8: Evaluate $\int e^{2x} \cos(3x) \, dx$.

  Solution: Let $u = e^{2x}, dv = \cos(3x) \, dx$. Then $du = 2e^{2x} \, dx$, and integrating dv gives $v = \int cos(3x) \, dx = \frac{1}{3} \sin(3x)$. Our integral becomes

$$uv - \int v \, du = \frac{1}{3} e^{2x} \sin(3x) - \frac{1}{3} \int \sin(3x) 2e^{2x} \, dx.$$

  Lets integrate $\int e^{2x} \sin(3x) \, dx$ by parts: Let $u = e^{2x}, dv = \sin(3x)$, then $du = 2e^{2x} \, dx$, and integrating dv gives $v = \int \sin(3x) \, dx = -\frac{1}{3} \cos(3x)$, and so

$$\int e^{2x} \sin(3x) \, dx = uv - \int v \, du = -\frac{1}{3} e^{2x} \cos(3x) + \frac{1}{3} \int \cos(3x) 2e^{2x} \, dx = -\frac{1}{3} e^{2x} \cos(3x) + \frac{2}{3} \int e^{2x} \cos(3x) \, dx.$$

  It looks like things are not getting better (cycling). But actually they are!! Let I represent the original integral $I = \int e^{2x} \cos(3x) \, dx$. We now have

$$I = \frac{1}{3} e^{2x} \sin(3x) - \frac{2}{3} [-\frac{1}{3} e^{2x} \cos(3x) + \frac{2}{3} I] = \frac{1}{3} e^{2x} \sin(3x) + \frac{2}{9} e^{2x} \cos(3x) - \frac{4}{9} I.$$

  Thus

$$I + \frac{4}{9} I = \frac{13}{9} I = \frac{1}{3} e^{2x} \sin(3x) + \frac{2}{9} e^{2x} \cos(3x)$$

  And so $I = \int e^{2x} \cos(3x) \, dx = \frac{9}{13} (\frac{1}{3} e^{2x} \sin(3x) + \frac{2}{9} e^{2x} \cos(3x))$.

- Example 9: Evaluate $\int \sec^3 x \, dx$. (Pam B's notes day 9, this is also an example of cycling)

- How do you know which order to write the functions in (see Example 4 above, where we had to change the order)? If one order doesnt work, try the other order.