Department of Mathematics, University of Houston
Math 1432
David Blecher, Some review questions for the Final

[Note that there is some duplication of questions below. For example, on the real final you'd have one of questions 8 and 9, not both, and similarly for some of the other questions.]

1. Let $g(x) = 5x^3 - 4x^2 + x - 2$. Find an equation for the normal line to the graph of g^{-1} at the point $(-2, 0)$.

2. (i) Find $h'(x)$ if $h(x) = x^x$, for $x > 0$.
   (ii) Find $\lim_{x\to\infty} x^x$ and $\lim_{x\to 0} x^x$.
   (iii) What is $\lim_{x\to\infty} (1 + \frac{2}{x})^x$?
   (iv) What is $\lim_{x\to 0} \frac{\sin x - x}{\cos x - 1}$.

3. Differentiate each of the following functions:
   (a) $\sec^{-1}(x^3)$. (b) $\ln(\cosh x)$. (c) $\log_4(1 + 2^x)$. (d) $(1 - \frac{1}{x})^x$.

4. Find the following integrals:
   (a) $\int \frac{1}{(x-1)(x^2-1)}\, dx$. (b) $\int_0^1 \tan^{-1} x\, dx$. (c) $\int e^{2x} \cos x\, dx$.
   (d) $\int \frac{1}{(4+x^2)^{\frac{3}{2}}}\, dx$. (e) $\displaystyle\int \frac{dx}{(4 - x^2)^{3/2}}$. (f) $\displaystyle\int_0^1 \frac{x^2}{x^6 + 1}\, dx$.

5. Find the following integral $\displaystyle\int_{-1}^{1} \frac{5x^3}{x^2 + x - 6}\, dx$.

6. $\displaystyle\int_1^5 \frac{x}{\sqrt{x - 1}}\, dx$. Does this converge? To what?

7. In each of the following write down if the series converges or diverges, giving reasons.
   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2 + 1}}$. (b) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k!}$. (c) $\displaystyle\sum_{k=1}^{\infty} (\sqrt{k^4 + 1} - \sqrt{k^4 - 1})$.
   (d) $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{k^k}$ (e) $\sum_{k=1}^{\infty} \frac{\sin k}{e^k}$ (f) $\sum_{k=1}^{\infty} \frac{(-1)^k \ln k}{k^2}$.

8. [Note that on the real final you'd probably only have to do (a) and (b) below.]
   (a) Find the **interval** of convergence of $\displaystyle\sum_{k=0}^{\infty} \frac{(k + 1)x^k}{2^k}$. Justify.
   (b) Find the integrated power series of the series in (a).
   (c) Show that the sum of the series in (b) is $\frac{2x}{2-x}$.
   (d) Set $f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(k + 1)x^k}{2^k}$ for x in the interval of convergence from (a). Show by differentiating the expression in (c), that $f(x) = \frac{4}{(2-x)^2}$ for x in the interval of convergence.
   (d) Setting $x = 1$ in (d) find $\sum_{k=1}^{\infty} \frac{k}{2^k}$.

9. [Note that on the real final you'd probably only have to do (a) and (b) below.]
   (a) For what values of x does the series $\sum_{k=0}^{\infty} \frac{x^k}{2^k}$ converge? For these values of x, what is the sum of the series?
   (b) Consider the series $\sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)2^k}$. What is its radius of convergence?
   (c) What is the sum of the series in (b), if $-1 < x < 1$? [Hint: integrate the expressions in (a).]
   (d) What is $\sum_{k=0}^{\infty} \frac{1}{(k+1)4^k}$? [Hint: choose x carefully in (c)].

10. Find the Taylor series for $f(x) = \ln(2 - x)$. Write down the first four terms out in full.

11. Using the Lagrange formula, find a Taylor series approximation for $\sin(1)$ with error < 0.001.

12. Give an equation of the tangent line to the curve given parametrically as $x(t) = \sin t, y(t) = \cos t$ at the point $(0, 1)$.

13. Sketch the polar curve $r = 1 - \cos\theta$, for $0 \le \theta \le 2\pi$.

14. Find the length of the curve in question 13.