Chapter 8. Integration techniques

8.1: Using integral tables.

Many integrals can be done simply by looking at the table of integrals on the course website. For example, $\int \frac{\sqrt{9-x^2}}{x^2}dx = -\frac{\sqrt{9-x^2}}{x} - \arcsin\frac{x}{3} + C$ by formula 90 in those tables. Sometimes of course we have to work with an integral a bit to get it into the form we see in the tables. For example, $\int \frac{\sqrt{9-4x^2}}{x^2}dx$. Let $u = 2x$ then $du = 2dx$, $x = u/2$ and $dx = \frac{1}{2}du$, and the integral becomes

$$\frac{1}{2}\int \frac{\sqrt{9-u^2}}{\frac{u^2}{4}}\,dx = 2\int \frac{\sqrt{9-u^2}}{u^2}du = -2\frac{\sqrt{9-u^2}}{u} - 2\arcsin\frac{u}{3} + C = -\frac{\sqrt{9-4x^2}}{x} - 2\arcsin\frac{2x}{3} + C.$$

8.2: Integration by parts.

The integration by parts formula is

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

To prove this note that $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$. Integrating, we get

$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx,$$

and so $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$.

- Example 1: Find $\int x\cos x\,dx$.

  Solution: $\int x\cos x\,dx = \int x\frac{d}{dx}(\sin x)\,dx = x\sin x - \int 1\cdot\sin x\,dx = x\sin x + \cos x$.

- Usually we write the integration by parts formula differently:

  $$\int u\,dv = uv - \int v\,du.$$

  [To see this is the same, let $u = f(x), v = g(x)$, then $du = f'(x)dx, dv = g'(x)dx$, and the last formula then reads $\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$.]

- Example 2: In this notation, evaluate $\int xe^x\,dx$.

  Solution: Let $u = x, dv = e^x\,dx$. Then $du = 1dx = dx$, and integrating dv gives $v = \int e^x\,dx = e^x$. So

  $$\int xe^x\,dx = \int u\,dv = uv - \int v\,du = xe^x - \int e^x dx = xe^x - e^x + C.$$

- Example 3: Find $\int_0^1 xe^x\,dx$.

  Solution: $[xe^x - e^x]_0^1 = e^1 - e^1 - [0 - e^0] = 1$.

- Example 4: Find $\int x^2\ln x\,dx$.

  Solution: The obvious choice here is $u = x^2$ and $dv = \ln x\,dx$. But this isnt so good because it will force us to integrate $\ln x$. Rather let us write $\int x^2\ln x\,dx = \int(\ln x)x^2\,dx$, and let $u = \ln x, dv = x^2\,dx$. Now $du = \frac{1}{x}dx$ and integrating dv gives $v = \int x^2\,dx = \frac{1}{3}x^3$. So

  $$\int(\ln x)x^2\,dx = \int u\,dv = uv - \int v\,du = (\ln x)\cdot\frac{1}{3}x^3 - \int\frac{1}{3}x^3\cdot\frac{1}{x}dx = \frac{x^3\ln x}{3} - \frac{1}{3}\int x^2dx,$$

  which is $\frac{x^3\ln x}{3} - \frac{1}{9}x^3 + C$.

- Example 5: Find $\int \ln x\, dx$.

  Solution: Let us write $\int \ln x\, dx = \int (\ln x) \cdot 1\, dx$, and let $u = \ln x$ and $dv = 1\, dx$. Then $du = \frac{1}{x} dx$ and integrating dv gives $v = x$, and so

$$\int \ln x\, dx = \int u\, dv = uv - \int v\, du = (\ln x) \cdot x - \int x \frac{1}{x} dx = x \ln x - \int 1 dx = x \ln x - x + C.$$

- Example 6: Evaluate $\int_0^1 \arcsin x\, dx$.

  Solution: Similar to the last: let $u = \arcsin x$, and $dv = dx$. Then $du = \frac{1}{\sqrt{1-x^2}} dx$, and integrating dv gives $v = x$, and so

$$\int u\, dv = uv - \int v\, du = x \arcsin x - \int x \frac{1}{\sqrt{1-x^2}} dx.$$

  Let $w = 1 - x^2$ then $dw = -2x dx$ and $2x dx = -\frac{1}{2} dw$. The integral becomes:

$$x \arcsin x + \frac{1}{2} \int \frac{1}{\sqrt{w}} dw = x \arcsin x + \frac{1}{2} \int w^{-\frac{1}{2}} dw = x \arcsin x + w^{\frac{1}{2}} = x \arcsin x + \sqrt{1-x^2}.$$

  Our final answer is $(x \arcsin x + \sqrt{1-x^2})]_0^1 = 1 \arcsin 1 - 0 - \sqrt{1} = \frac{\pi}{2} - 1$.

- Example 7: Evaluate $\int_0^1 x^2 e^x\, dx$.

  Solution: Let $u = x^2, dv = e^x\, dx$ then $du = 2x dx, v = e^x$, and our integral becomes

$$uv - \int v\, du = x^2 e^x - 2 \int e^x x dx.$$

  We did $\int x e^x dx$ earlier, and so our integral becomes

$$(x^2 e^x - 2(x e^x - e^x))]_0^1 = e - 2(e - e) - (-2(-1)) = e - 2.$$

- Example 8: Evaluate $\int e^{2x} \cos(3x)\, dx$.

  Solution: Let $u = e^{2x}, dv = \cos(3x)\, dx$. Then $du = 2e^{2x}\, dx$, and integrating dv gives $v = \int cos(3x)\, dx = \frac{1}{3} \sin(3x)$. Our integral becomes

$$uv - \int v\, du = \frac{1}{3} e^{2x} \sin(3x) - \frac{1}{3} \int \sin(3x) 2e^{2x}\, dx.$$

  Lets integrate $\int e^{2x} \sin(3x)\, dx$ by parts: Let $u = e^{2x}, dv = \sin(3x)$, then $du = 2e^{2x}\, dx$, and integrating dv gives $v = \int \sin(3x)\, dx = -\frac{1}{3} \cos(3x)$, and so

$$\int e^{2x} \sin(3x)\, dx = uv - \int v\, du = -\frac{1}{3} e^{2x} \cos(3x) + \frac{1}{3} \int \cos(3x) 2e^{2x}\, dx = -\frac{1}{3} e^{2x} \cos(3x) + \frac{2}{3} \int e^{2x} \cos(3x)\, dx.$$

  It looks like things are not getting better (cycling). But actually they are!! Let I represent the original integral $I = \int e^{2x} \cos(3x)\, dx$. We now have

$$I = \frac{1}{3} e^{2x} \sin(3x) - \frac{2}{3}[-\frac{1}{3} e^{2x} \cos(3x) + \frac{2}{3} I] = \frac{1}{3} e^{2x} \sin(3x) + \frac{2}{9} e^{2x} \cos(3x) - \frac{4}{9} I.$$

  Thus

$$I + \frac{4}{9} I = \frac{13}{9} I = \frac{1}{3} e^{2x} \sin(3x) + \frac{2}{9} e^{2x} \cos(3x)$$

  And so $I = \int e^{2x} \cos(3x)\, dx = \frac{9}{13}(\frac{1}{3} e^{2x} \sin(3x) + \frac{2}{9} e^{2x} \cos(3x))$.

- Example 9: Evaluate $\int \sec^3 x\, dx$. (done in the next section as an example of cycling)

- How do you know which order to write the functions in (see Example 4 above, where we had to change the order)? If one order doesnt work, try the other order.

8.3: Integrals of products of trig functions.

- The first kind of product we look at, will need the trig double angle formulae:

$$\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$
$$\sin A \cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$$
$$\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$$

- Example: Evaluate $\int \sin(5x)\sin(3x)\,dx$.

  Solution: $= \frac{1}{2}\int(\cos(2x) - \cos(8x))\,dx = \frac{1}{4}\sin(2x) - \frac{1}{16}\sin(8x) + C$.

- Next we look at integrals of form $\int \sin^n x\,dx$ and $\int \cos^n x\,dx$. When $n = 1$ these are just $-\cos x + C$ and $\sin x + C$. When $n = 2$ we give two methods:

- Find $\int \sin^2 x\,dx$.

  Solution: Method 1: use the trig double angle formula $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, which follows from the facts $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$. So

$$\int \sin^2 x\,dx = \frac{1}{2}\int(1-\cos 2x)\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C.$$

  Method 2: Integration by parts: $\int \sin^2 x\,dx = \int \sin x \sin x\,dx$, and if we set $u = \sin x$ and $dv = \sin x\,dx$ then $du = \cos x dx$ and $v = \int \sin x\,dx = -\cos x$, and our integral becomes

$$uv - \int vdu = -\sin x\cos x + \int \cos x\cos x\,dx = -\frac{1}{2}\sin 2x + \int(1-\sin^2 x)\,dx = -\frac{1}{2}\sin 2x + x - I$$

  where $I = \int \sin^2 x\,dx$. So $2I = -\frac{1}{2}\sin 2x + x$, and $I = -\frac{1}{4}\sin 2x + \frac{x}{2}$.

- More generally, we apply the last method to find $\int \sin^n x\,dx$. We have $\int \sin^n x\,dx = \int(\sin x)^{n-1}\sin x\,dx$. Set $u = \sin^{n-1} x$ and $dv = \sin x\,dx$ then $du = (n-1)\sin^{n-2} x\cos x dx$ and $v = \int \sin x\,dx = -\cos x$, and our integral becomes

$$uv - \int vdu = -\sin^{n-1} x\cos x + \int \cos x(n-1)\sin^{n-2} x\cos x\,dx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos^2 x\,dx,$$

  Writing $\cos^2 x = 1 - \sin^2 x$, our integral becomes

$$I_n = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx = -\sin^{n-1} x\cos x + (n-1)I_{n-2} - (n-1)I_n,$$

  where $I_n = \int \sin^n x\,dx$. Hence $I_n + (n-1)I_n = nI_n = -\sin^{n-1} x\cos x + (n-1)I_{n-2}$, so

$$I_n = -\frac{1}{n}\sin^{n-1} x\,\cos x + \frac{n-1}{n}I_{n-2}.$$

  This is called a *reduction formula.*

- Note that if our integral is $\int_0^{\frac{\pi}{2}}$, then $-\frac{1}{n}\sin^{n-1} x\cos x]_0^{\frac{\pi}{2}} = 0 - 0 = 0$, and so the last reduction formula becomes

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x\,dx.$$

- Example. Find $\int_0^{\frac{\pi}{2}} \sin^6 x\,dx$.

  Solution: $= \frac{5}{6}\int_0^{\frac{\pi}{2}} \sin^4 x\,dx = \frac{5}{6}\frac{3}{4}\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{5}{6}\frac{3}{4}\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^0 x\,dx = \frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2}$.

- By the same idea, $\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{3}{4}\frac{1}{2}\frac{\pi}{2}$ if n is even, and $\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{4}{5}\frac{2}{3}$ if n is odd.

- There are similar formula for $J_n = \int \cos^n x\, dx$ (see text).

$$J_n = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} J_{n-2}.$$

In particular,

$$\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \cos^{n-2} x\, dx,$$

etc, as in the case of sin. Similarly, there are similar formula for $\int \tan^n x\, dx$ and $\int \sec^n x\, dx$ (see text).

- We now look at integrals of the form

$$\int \sin^n x \cos^m x\, dx$$

RULE: If n odd, substitute $u = \cos x$. If m odd, substitute $u = \sin x$. If both are even use $sin^2 + \cos^2 = 1$, and the reduction formula for $I_n = \int \sin^n x\, dx$ and $J_n = \int \cos^n x\, dx$ above.

- Example. Find $\int \sin^3 x \cos^2 x\, dx$.

Solution: Let $u = \cos x$ then $du = -\sin x dx$ and our integral becomes

$$-\int u^2 \sin^2 x\, du = -\int u^2 (1-\cos^2 x)\, du = -\int u^2(1-u^2)\, du = -\frac{u^3}{3} + \frac{u^5}{5} + C = -\frac{\cos^3 x}{3} + \frac{cos^5 x}{5} + C.$$

- Example. Find $\int \sin^4 x \cos^2 x\, dx$.

Solution: We rewite the cos terms here in terms of sin. We get

$$\int \sin^4 x \cos^2 x\, dx = \int \sin^4 x\,(1 - \sin^2 x)\, dx = I_4 - I_6,$$

where $I_n = \int \sin^n x\, dx$. Now by the reduction formula we have $I_6 = -\frac{1}{6}\sin^5 x \cos x + \frac{5}{6} I_4$. So $I_4 - I_6 = \frac{1}{6}(I_4 + \sin^5 x \cos x)$. By the reduction formula again,

$$I_4 = -\frac{1}{4}\sin^3 x \cos x + \frac{3}{4} I_2 = -\frac{1}{4}\sin^3 x \cos x + \frac{3}{4}(-\frac{1}{2}\sin x \cos x + \frac{x}{2}) = -\frac{1}{4}\sin^3 x \cos x - \frac{3}{8}\sin x \cos x + \frac{3x}{8}.$$

Putting this together, our final answer is $\frac{1}{6}(I_4 + \sin^5 x \cos x)$, which is

$$\frac{1}{6}(-\frac{1}{4}\sin^3 x \cos x - \frac{3}{8}\sin x \cos x + \frac{3x}{8} + \sin^5 x \cos x) + C.$$

(An alternative way to do even powers of sin and cos is to use double angle formula, so for example $\int \sin^4 x \cos^2 x\, dx = \int (\sin x \cos x)^2 \sin^2 x dx$, and then use $\sin x \cos x = \frac{1}{2}\sin(2x)$ and $\sin^2 x = \frac{1-\cos(2x)}{2}$, and the integral becomes $\frac{1}{8}\int \sin^2(2x)\,(1 - \cos(2x))\, dx$, which is a simpler integral than the one we started with (sometimes one has to repeat this process and apply the double angle formula again, and so on. We'll do another example of this below.))

- Example. Find $\int \sin^4 x \cos^6 x\, dx$.

Solution: We rewite the cos terms here in terms of sin. We get

$$\int \sin^4 x\,(\cos^2 x)^3\, dx = \int \sin^4 x\,(1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x)\, dx,$$

since $(1-t)^3 = 1 - 3t + 3t^2 - t^3$. So our integral becomes

$$\int \sin^4 x\,dx - 3\int \sin^6 x\,dx + 3\int \sin^8 x\,dx - \int \sin^{10} x\,dx = I_4 - 3I_6 + 3I_8 - I_{10}.$$

Here $I_n = \int \sin^n x\,dx$. However one may compute I_4, I_6, I_8, I_{10} as in the last example by the reduction formula.

- Example. Find $\int \sin^5 x\,dx$.

  Solution: This is I_5, so one can do it by the reduction formula as above. Or one may write $\int \sin^5 x\,dx = \int \sin^4 x \sin x dx = \int (1-\cos^2 x)^2 \sin x dx = -\int (1-u^2)^2 du$ by the u-substitution $u = \cos x$. We get

  $$-\int (1 - 2u^2 + u^4)du = -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C.$$

- Example. Find $\int \cos^4 x\,dx$.

  Solution: This could be done by the reduction formula for $J_4 = \int \cos^4 x\,dx$ in the text book. Alternatively, by a trig double angle formula, $\cos^4 x = (\cos^2 x)^2 = (\frac{1+\cos 2x}{2})^2 = \frac{1+2\cos 2x+\cos^2 2x}{4}$. Moreover by the same trig double angle formula, $\cos^2 2x = \frac{1+\cos 4x}{2}$. So our integral becomes

  $$\frac{1}{4}\int 1dx + \frac{1}{2}\int \cos 2x\,dx + \frac{1}{4}\int \cos^2 2x\,dx = \frac{x}{4} - \frac{\sin 2x}{4} + \frac{1}{8}\int (1+\cos 4x)dx.$$

  Final answer: $\frac{x}{4} - \frac{\sin 2x}{4} + \frac{x}{8} + \frac{\sin 4x}{32} + C$.

- The RULE above (if n odd, substitute $u = \cos x$, if m odd, substitute $u = \sin x$, etc) usually works for a power of sin divided by a power of cos (or vice versa). For example:

- Example. Find $\int \frac{\sin^3 x}{\cos^8 x}\,dx$.

  Solution: Let $u = \cos x$ then $du = -\sin x\,dx$, so our integral becomes

$$\int \frac{\sin^2 x}{u^8} \sin x\,dx = -\int \frac{1-u^2}{u^8}\,du = -\int (u^{-8} - u^{-6})\,du = \frac{u^{-7}}{7} - \frac{u^{-5}}{5} + C = \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C.$$

- To integrate products of powers of other trig functions, one can often simply write these other trig functions in terms of sines and cosines:

- Example. Find $\int \tan^3 x \sec^5 x\,dx$.

  Solution: $= \int \frac{\sin^3 x}{\cos^3 x}\frac{1}{\cos^5 x}\,dx = \int \frac{\sin^3 x}{\cos^8 x}\,dx = \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C$.

- Example. Find $\int \tan^5 x \sec^2 x\,dx$.

  Solution: Let $u = \tan x$ then $du = \sec^2 x\,dx$ so our integral becomes

  $$\int u^5 du = \frac{u^6}{6} + C = \frac{\tan^6 x}{6} + C.$$

- The technique in the last example works whenever we have $\int \tan^n x \sec^2 x\,dx$ (just let $u = \tan x$ and the integral becomes $\int u^n du$). If you see something like $\int \tan^n x \sec^4 x\,dx$ for example, write this as $\int \tan^n x\,(\tan^2 + 1)\sec^2 x\,dx$ (and again let $u = \tan x$, then the integral becomes $\int u^n(u^2+1)du$.)

- Example. Find $\int \sec^4 x\,dx$.

Solution: $= \int \sec^2 x \sec^2 x\, dx = \int \sec^2 x(\tan^2 x + 1)\, dx$. If we let $u = \tan x$ then $du = \sec^2 x\, dx$ and our integral becomes

$$\int (u^2+1)du + C = \frac{u^3}{3} + u + C = \frac{\tan^3 x}{3} + \tan x + C.$$

(This example is just the case $n = 0$ of the line before the Example.)

- Example. Find $\int \sec^3 x\, dx$.

  Solution: We do $I = \int \sec x \sec^2 x\, dx$ by parts:

$$= \sec x\, \tan x - \int \tan x\, \sec x\, \tan x\, dx = \sec x\, \tan x - \int (\sec^2 x - 1) \sec x\, dx = \sec x\, \tan x - I + \int \sec x\, dx.$$

  This is an example of 'cycling'. So $2I = \sec x\, \tan x + \ln|\sec x + \tan x|$, and $I = \frac{1}{2}(\sec x\, \tan x + \ln|\sec x + \tan x|) + C$.

- Example. Find $\int \tan^4 x\, dx$.

  Solution: $\int \tan^4 x\, dx = \int \tan^2 x(\sec^2 x - 1)\, dx = \int \tan^2 x \sec^2 x - \int \tan^2 x\, dx = \int u^2 du - \int (\sec^2 x - 1)\, dx = \frac{u^3}{3} - \tan x + x = \frac{\tan^3 x}{3} - \tan x + x$.

- Example. Find $\int \tan^6 x\, dx$.

  Solution: $= \int \tan^4 x(\sec^2 x - 1)\, dx = \int \tan^4 x \sec^2 x\, dx - \int \tan^4 x\, dx$. If we let $u = \tan x$ then $du = \sec^2 x\, dx$ and $\int \tan^4 x \sec^2 x\, dx = \int u^4 du = \frac{u^5}{5} = \frac{\tan^5 x}{5}$. Also, $\int \tan^4 x\, dx = \frac{\tan^3 x}{3} - \tan x + x$ (last example). Final answer: $\frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + C$.

8.4: Trig substitutions.

- RULE: For an integral with term $\sqrt{a^2 - x^2}$ substitute $x = a \sin\theta$. With term $x^2 + a^2$ substitute $x = a\tan\theta$. With term $\sqrt{x^2 - a^2}$ substitute $x = a \sec\theta$.

- Example. Find $\int_0^2 \sqrt{4 - x^2}\, dx$.

  Solution: Let $x = 2\sin\theta$ then $dx = 2\cos\theta d\theta$ and our integral becomes

$$\int \sqrt{4 - 4\sin^2\theta}\, 2\cos\theta d\theta = 4\int \cos^2\theta d\theta = 4(\frac{\theta}{2} + \frac{\sin(2\theta)}{4})$$

  (we did $\int \cos^2 x\, dx$ in a previous section, using a trig double angle formula). When $x = 0$ then $\theta = 0$ and when $x = 2$ then $\theta = \frac{\pi}{2}$ (since $2\sin\frac{\pi}{2} = 2$). So our integral is

$$(2\theta + \sin(2\theta))]_0^{\frac{\pi}{2}} = 2\frac{\pi}{2} - 0 = \pi.$$

- Example. Find $\int \frac{dx}{(9+4x^2)^{\frac{3}{2}}}$.

  Solution: We write this as $\int \frac{dx}{(9+(2x)^2)^{\frac{3}{2}}}$, and let $2x = 3\tan\theta$. So $2dx = 3\sec^2\theta d\theta$, so $dx = \frac{3}{2}\sec^2\theta d\theta$, and our integral becomes

$$\frac{3}{2}\int \frac{\sec^2\theta}{(9+(3\tan\theta)^2)^{\frac{3}{2}}} d\theta = \frac{3}{2}\int \frac{\sec^2\theta}{9^{\frac{3}{2}}(1+\tan^2\theta)^{\frac{3}{2}}} d\theta = \frac{1}{18}\int \frac{\sec^2\theta}{\sec^3\theta} d\theta = \frac{1}{18}\int \cos\theta d\theta = \frac{1}{18}\sin\theta + C.$$

  To find $\sin\theta$ in terms of x we draw a right triangle. Since $2x = 3\tan\theta$ we have $\tan\theta = \frac{2x}{3} = \frac{\text{opp}}{\text{adj}}$. So draw a right triangle with opp $= 2x$, adj $= 3$. By Pythagoras the hypotenuse is $\sqrt{9+4x^2}$, so $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{2x}{\sqrt{9+4x^2}}$. Final answer: $\frac{1}{18}\frac{2x}{\sqrt{9+4x^2}} + C = \frac{x}{9\sqrt{9+4x^2}} + C$.

- Example. Find $\int \frac{dx}{\sqrt{x^2-4x-5}}\,dx$.

  Solution: We complete the square: $x^2-4x-5=(x-2)^2-9$. Our integral becomes $\int \frac{dx}{\sqrt{(x-2)^2-3^2}}$. Let $x-2=3\sec\theta$. So $dx=3\tan\theta\,\sec\theta d\theta$, and our integral becomes

$$\int \frac{3\tan\theta\ \sec\theta}{\sqrt{(3\sec\theta)^2-9}}d\theta=\frac{3}{3}\int\frac{\tan\theta\ \sec\theta}{\tan\theta}d\theta=\int\sec\theta d\theta=\ln|\sec\theta+\tan\theta|+C.$$

  We know $\sec\theta=\frac{x-2}{3}$. To find $\tan\theta$ in terms of x we draw a right triangle. Since $\sec\theta=\frac{x-2}{3}=\frac{\text{hyp}}{\text{adj}}$, we draw a right triangle with hypotenuse $=x-2$ and adj $=3$. By Pythagoras the opp $=\sqrt{(x-2)^2-9}=\sqrt{x^2-4x-5}$, so $\tan\theta=\frac{\text{opp}}{\text{adj}}=\frac{\sqrt{x^2-4x-5}}{3}$. Final answer: $\ln|\frac{x-2}{3}+\frac{\sqrt{x^2-4x-5}}{3}|+C$.

- Example. Find $\int \frac{dx}{e^x\sqrt{e^{2x}-9}}\,dx$ and $\int \frac{x^2}{\sqrt{4+x^2}}dx$.

  Solution: Done in class (if you did not write down the details in class you can also find them in e.g. Pam B's online notes day 13).