Math 1432

REVIEW

The one posted plus some extra problems that were included in previous semesters, noted with a (*).

If you find any errors, please let me know. Thanks.

1. Give the equation of the tangent line to the given graph at the point where x = 0

a. $f(x) = ln(6x+1) + e^{2x}$

$f'(x) = \frac{6}{6x+1} + 2e^{2x}$

$f(0) = 1$

$f'(0) = 6 + 2 = 8$

$\boxed{TL: \ y - 1 = 8(x - 0)}$

b. $f(x) = ln(2x+1) - 3e^{-4x}$

$f'(x) = \frac{2}{2x+1} + 12e^{-4x}$

$f(0) = -3$

$f'(0) = 2 + 12 = 14$

$\boxed{TL: \ y + 3 = 14(x - 0)}$

c. $f(x) = \sqrt{9 - x^2}$

$f'(x) = \frac{1}{2}(9 - x^2)^{-1/2}(-2x)$

$f(0) = 3$

$f'(0) = 0$

$\boxed{TL: \ y = 3}$

2. Find the inverse of the following, if possible:

a. $f(x)=\dfrac{2}{3-x}$

$x=\dfrac{2}{3-y}$

$f'(x)=\dfrac{2}{(3-x)^2}>0$

$\therefore$ invertible

$3x-xy=2$

$3x-2=xy$

$\boxed{\dfrac{3x-2}{x}=y=f^{-1}(x)}$

b. $f(x)=\dfrac{x+1}{x+2}$

$x=\dfrac{y+1}{y+2}$

$f'(x)=\dfrac{x+2-x-1}{(x+2)^2}$

$=\dfrac{1}{(x+2)^2}>0$

$\therefore$ invertible

$xy+2x=y+1$

$xy-y=1-2x$

$\boxed{y=\dfrac{1-2x}{x-1}=f^{-1}(x)}$

3. Find the derivative of the inverse for the following:

$$\boxed{\begin{array}{l} f(a)=b \\ f^{-1}(b)=a \\ (f^{-1})'(b)=\dfrac{1}{f'(a)} \end{array}}$$

a. $f(x)=x^3+1,\ \ f(2)=9,\ \ (f^{-1})'(9)=$

$f'(x)=3x^2$

$(f^{-1})'(9)=\dfrac{1}{f'(2)}=\boxed{\dfrac{1}{12}}$

b. $f(-3)=1, f(1)=2, f'(-3)=3, f'(1)=-2,\ \ (f^{-1})'(1)=$

$(f^{-1})'(1)=\dfrac{1}{f'(-3)}=\boxed{\dfrac{1}{3}}$

c. $f(x)$ passes through the points (3 , –2) and (–2 , 1). The slope of the tangent line to the graph of $f(x)$ at $\boldsymbol{x}$ = 3 is –1/4. Evaluate the derivative of the inverse of f at –2.

$f'(3)=-\frac{1}{4}$

$(f^{-1})'(-2)=\dfrac{1}{f'(3)}=\dfrac{1}{-\frac{1}{4}}=\boxed{-4}$

4. Find the equation of the tangent and the normal lines to the parametric curves at the given points:

a. $x(t) = -2\cos 2t, y(t) = 4 + 2t, (-2,4)$

$\frac{dy}{dx} = \frac{2}{4\sin 2t}\Big|_{t=0} = \text{undef.}$

$-2 = -2\cos 2t$
$t = 0$

$4 = 4 + 2t$
$t = 0$

$\boxed{\therefore \quad TL: X = -2 \quad NL: y = 4}$

b. $x(t) = 3\cos(3t) + 2t, y(t) = 1 + 5t, (3,1)$

$\frac{dy}{dx} = \frac{5}{-9\sin 3t + 2}\Big|_{t=0} = \frac{5}{2}$

$3 = 3\cos 3t$
$t = 0$

$1 = 1 + 5t$
$t = 0$

$$\boxed{\begin{array}{l} TL: y - 1 = \frac{5}{2}(x-3) \\ NL: y - 1 = -\frac{2}{5}(x-3) \end{array}}$$

5. Give an equation relating x and y for the curve given parametrically by

a. $x(t) = -1 + 3\cos t \quad y(t) = 1 + 2\sin t$

$\frac{x+1}{3} = \cos t \qquad \frac{y-1}{2} = \sin t$

$\left(\frac{x+1}{3}\right)^2 + \left(\frac{y-1}{2}\right)^2 = 1$

b. $x(t) = -1 + 3\cosh t \quad y(t) = 1 + 2\sinh t$

$\frac{x+1}{3} = \cosh t \qquad \frac{y-1}{2} = \sinh t$

$\left(\frac{x+1}{3}\right)^2 - \left(\frac{y-1}{2}\right)^2 = 1$

c. $x(t) = -1 + 4e^t \quad y(t) = 2 + 3e^{-t}$

$\frac{x+1}{4} = e^t \qquad \frac{y-2}{3} = e^{-t}$

$e^t = \frac{3}{y-2}$

$\frac{x+1}{4} = \frac{3}{y-2}$

$y - 2 = \frac{12}{x+1}$

$\boxed{y = \frac{12}{x+1} + 2}$

6. Differentiate the function:

a. $f(x)=3^{x^2}$

$f'(x) = 3^{x^2} \cdot \ln 3 \cdot 2x$

Note: $\log_5 x = \frac{\ln x}{\ln 5}$

b. $f(x)=\tan(\log_5 x)$

$f'(x) = \sec^2(\log_5 x) \cdot \frac{1}{\ln 5} \cdot \frac{1}{x}$

c. $f(x)=x^{\sin x}$

$\ln y = \ln x^{\sin x}$

$= (\sin x)(\ln x)$

$\frac{dy}{y} = \frac{\sin x}{x} + (\cos x)\ln x$

$dy = x^{\sin x}\left(\frac{\sin x}{x} + (\cos x)\ln x\right)$

d. $f(x)=\sinh(3x)$

$f'(x) = 3\cosh(3x)$

e. $f(x)=\frac{\cosh x}{x}$ $\quad f'(x) = \frac{x\sinh x - \cosh x}{x^2}$

7. Integrate:

a. $\int(\cosh(3x)+\sinh(2x))dx = \frac{1}{3}\sinh(3x) + \frac{1}{2}\cosh(2x) + c$

b. $\int 4^{3x}\,dx = \frac{4^{3x}}{3\ln 4} + C$

c. $\int \frac{\log_2(x^3)}{x}dx = \int \frac{\ln x^3}{\ln 2} \cdot \frac{1}{x}\,dx = \frac{3}{\ln 2}\int \frac{\ln x}{x}\,dx$

$= \frac{3}{2\ln 2} \cdot (\ln x)^2 + C$

d. $\int\left(2^{7x} - \sinh(5x)\right)dx$

$= \frac{2^{7x}}{7\ln 2} - \frac{1}{5}\cosh(5x) + C$

e. $\int \frac{\sin(3x)}{16+\cos^2(3x)}dx = \frac{-1}{3}\int \frac{-3\sin 3x}{16+\cos^2(3x)}dx$

$u = \cos 3x \quad a=4 \quad = -\frac{1}{3}\cdot\frac{1}{4}\arctan\frac{\cos 3x}{4} + C$

$du = -3\sin 3x$

f. $\int \frac{6x}{4+x^4}dx = 3\int \frac{2x}{4+x^4}dx = 3\cdot\frac{1}{2}\arctan\frac{x^2}{2} + C$

$u = x^2 \quad a = 2$

$du = 2x\,dx$

g. $\int \tan(3x)dx = -\frac{1}{3}\ln|\cos(3x)| + C$

h. $\int \frac{\arctan(3x)}{1+9x^2}dx = \frac{1}{3}\int \frac{3\arctan(3x)}{1+9x^2}dx$

$= \frac{1}{3}\cdot\frac{1}{2}(\arctan 3x)^2 + C$

i. $\int \frac{1}{\sqrt{4+x^2}}dx = \int \frac{2\sec^2\theta\, d\theta}{2\sec\theta} = \int \sec\theta\, d\theta$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$\sqrt{4+x^2} = 2\sec\theta$

$= \ln|\sec\theta + \tan\theta| + C$

$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C$

$= \ln\left|\sqrt{4+x^2} + x\right| + C$

j. $\int \sqrt{9-x^2}\,dx = \int 3\cos\theta\cdot 3\cos\theta\, d\theta$

$x = 3\sin\theta$

$dx = 3\cos\theta$

$\sqrt{9-x^2} = 3\cos\theta$

$= 9\int \cos^2\theta\, d\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

$= \frac{9}{2}\int (1+\cos 2\theta)d\theta$

$= \frac{9}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$

$= \frac{9}{2}\left(\arcsin\frac{x}{3} + \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C$

k. $\int 3\ln(4x)dx$

$= 3\int(\ln 4 + \ln x)dx$

$= (3\ln 4)x + 3(x\ln x - x) + C$

l. $\int x^2 e^x dx = x^2e^x - 2xe^x + 2e^x + C$

$x^2 \quad e^x \quad +$
$2x \quad e^x \quad -$
$2 \quad e^x \quad +$
$\quad e^x \quad -$
$\quad\quad +$

m. $\int \frac{5x+14}{(x+1)(x^2-4)}dx = \int\left(\frac{-3}{x+1} + \frac{1}{x+2} + \frac{2}{x-2}\right)dx$

$\frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x-2}$ $\quad = -3\ln|x+1| + \ln|x+2| + 2\ln|x-2| + C$

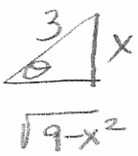
$x=2 \Rightarrow C=2$
$x=-2 \Rightarrow B=1$
$x=-1 \Rightarrow A=-3$

n. $\int \frac{x^2+5x+2}{(x+1)(x^2+1)}dx = \int \frac{-1}{x+1} + \frac{2x}{x^2+1} + \frac{3}{x^2+1}\,dx$

$\frac{A}{x+1} + \frac{Bx+C}{x^2+1}$ $\quad = -1\ln|x+1| + \ln|x^2+1| + 3\arctan x + C$

$x=-1 \Rightarrow A=-1$
$x=0 \Rightarrow C=3$
$x=1 \Rightarrow B=2$

o. $\int \frac{2x^2}{\sqrt{9-x^2}}dx = 2\int \frac{9\sin^2\theta \; \cancel{3\cos\theta}\, d\theta}{\cancel{3\cos\theta}}$



$x = 3\sin\theta$
$x^2 = 9\sin^2\theta$
$\sqrt{9-x^2} = 3\cos\theta$
$dx = 3\cos\theta\, d\theta$

$= 18\int \sin^2\theta\, d\theta$
$= 9\int(1-\cos 2\theta)d\theta$
$= 9\left(\theta - \frac{\sin 2\theta}{2}\right) + C$
$= 9(\theta - \sin\theta\cos\theta) + C$
$= 9\left(\arcsin\frac{x}{3} - \frac{x}{3}\frac{\sqrt{9-x^2}}{3}\right) + C$

p. $\int 2\arctan(10x)dx = 2\left[x\arctan 10x - \int \frac{10x}{1+100x^2}dx\right]$

$u = \arctan 10x \quad dv = dx$
$du = \frac{10}{1+100x^2}dx \quad v = x$

$= 2\left[x\arctan 10x - \frac{1}{20}\ln|1+100x^2|\right] + C$
$= 2x\arctan 10x - \frac{1}{10}\ln|1+100x^2| + C$

q. $\int 3x\cos(2x)dx = \frac{3x\sin 2x}{2} + \frac{3\cos 2x}{4} + C$

$3x \quad \cos 2x \quad +$
$3 \quad \frac{\sin 2x}{2} \quad -$
$\quad -\frac{\cos 2x}{4} \quad +$
$\quad\quad -$

8. Write an expression for the nth term of the sequence:

a. 1, 4, 7, 10, ...

$a_n = 1 + 3(n-1)$
$= -2 + 3n$

b. $2, -1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \ldots$

$a_n = 2\left(-\frac{1}{2}\right)^{n-1}$

9. Determine if the following sequences are monotonic. Also indicate if the sequence is bounded and if it is give the least upper bound and/or greatest lower bound.

a. $a_n = \dfrac{2n}{1+n}$

Bounded;
n starts with 1

$f(x) = \frac{2x}{1+x}$

$f'(x) = \frac{(1+x)2 - 2x}{(1+x)^2} > 0$

$\therefore$ monotonic; increasing

GLB = 0
LUB = 2

b. $a_n = \dfrac{\cos n}{n}$

n starts with 1

not monotonic; $-1 \le \cos n \le 1$
bounded. $\left[\frac{\cos 3}{3}, \frac{\cos 1}{1}\right]$

least upper bound

cos 3 almost cos π

10. Determine if the following sequences converge or diverge. If they converge, give the limit.

a. $\left\{(-1)^n\left(\dfrac{n}{n+1}\right)\right\}$ d'ge

b. $\left\{\dfrac{6n^2 - 2n + 1}{4n^2 - 1}\right\} \rightarrow \frac{6}{4}$ c'ges

c. $\left\{\dfrac{(n+2)!}{n!}\right\}$ d'ges

d. $\left\{\dfrac{3}{e^n}\right\} \rightarrow 0$ c'ges

e. $\left\{\dfrac{4n+1}{n^2-3n}\right\} \to 0$ c'ges

f. $\left\{\dfrac{e^n}{n^3}\right\} \to \infty$ d'ges

*g. $\left\{\dfrac{2n^2+1}{3n^3+4n^2+6}\right\} \to 0$ c'ges

*h. $\left\{\dfrac{1}{n\,ln(n)}\right\} \to 0$ c'ges

*i. $\{n\,sin(1/n)\} \to 1$ c'ges $\dfrac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$

*j. $\left\{\left(\dfrac{n-1}{n}\right)^n\right\} = \left\{\left(1-\frac{1}{n}\right)^n\right\} \to e^{-1}$ c'ges

11. Determine if the following series (A) converge absolutely, (B) converge conditionally or (C) diverge.

a. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+3}$

$\sum \frac{\sqrt{n}}{n+3}$ compare to $\frac{1}{n^{1/2}}$ d'ges

Alt. Ser. conv. $\therefore$ (B)

b. $\sum_{n=1}^{\infty} \frac{\cos \pi n}{n^2}$

$\sum \frac{1}{n^2}$ (A)

c. $\sum_{n=0}^{\infty} \frac{4n(-1)^n}{3n^2+2n+1}$

$\sum \frac{4n}{3n^2+2n+1}$ compare $\frac{1}{n}$ d'ges

Alt. ser conv. (B)

Alt. Series test:
① $\lim_{n\to\infty} b_n = 0$
② $b_{n+1} < b_n$

d. $\sum_{n=0}^{\infty} \frac{3(-1)^n}{\sqrt{3n^2+2n+1}}$

$\sum \frac{3}{\sqrt{3n^2+2n+1}}$ compare $\frac{1}{n}$, d'ges

Alt. ser. conv. (B)

e. $\sum_{n=0}^{\infty} \frac{3n(-1)^n}{\sqrt{3n^2+2n+1}}$

$\sum \frac{3n}{\sqrt{3n^2+2n+1}}$ d'ges BDT

$\lim_{n\to\infty} \frac{3n}{\sqrt{3n^2+2n+1}} \neq 0$ d'ges (C)

f. $\sum_{n=0}^{\infty} \left(4(-1)^n \left(\frac{n}{n+3} \right)^n \right)$

$\sum 4\left(\frac{n+3}{n}\right)^{-n}$ d'ges $\to e^{-3}$

d'ges (C) $\lim_{n\to\infty} \neq 0$

g. $\sum_{n=0}^{\infty} \left(\frac{2(-1)^n \arctan n}{3+n^2+n^3} \right)$

$\sum \frac{2\arctan n}{3+n^2+n^3}$ compare $\frac{\pi}{n^3}$ c'ges (A)

h. $\sum_{n=0}^{\infty}\left(\frac{(-1)^n 3^n}{4^n+3n}\right)$ $\sum \frac{3^n}{4^n+3n}$ $\left(\frac{3}{4}\right)^n$ c'ges A

i. $\sum_{n=0}^{\infty}\left(\frac{(-1)^n 3}{(n+2)\ln(n+2)}\right)$ $\sum \frac{3}{(n+2)\ln(n+2)}$ Alt. ser. conv. B

Integral test
cont, pos, dec

$3\int_0^{\infty}\frac{1}{(x+2)\ln(x+2)}dx$

d'ges

*j. $\sum_{n=2}^{\infty}\frac{(-1)^n n!}{(n+1)!}$ $\sum \frac{n!}{(n+1)!}=\sum \frac{1}{n+1}$ d'ges Alt. ser. conv B

*k. $\sum_{n=2}^{\infty}\frac{(-1)^n}{3n+2}$ $\sum \frac{1}{3n+2}$ d'ges Alt. ser. conv B

*l. $\sum_{n=0}^{\infty}\frac{(-1)^n 10n^2}{3^n}$ $\sum \frac{10n^2}{3^n}$ c'ges A

Ratio test

*m. $\sum_{n=2}^{\infty}\frac{(-1)^n 3^n}{n!}$ $\sum \frac{3^n}{n!}$ c'ges A

Ratio test

*n. $\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+3n+2}$ $\sum \frac{1}{n^2+3n+2}$ c'ges A

compare $\frac{1}{n^2}$

*o. $\sum_{n=2}^{\infty}\frac{\cos(\pi n)n^n}{n!}$ $\sum \frac{n^n}{n!}\to\infty$ d'ges C

12. Find the sum of the following convergent series:

a. $\sum_{n=0}^{\infty} 2\left(-\frac{4}{9}\right)^n$ $\quad S = \frac{2}{1-\left(-\frac{4}{9}\right)} = \frac{2}{\frac{13}{9}} = \frac{18}{13}$

b. $\sum_{n=0}^{\infty}\left(\frac{1}{3^n}-\frac{5}{6^n}\right)$ $\quad S = \frac{1}{1-\frac{1}{3}} - \frac{5}{1-\frac{1}{6}}$

$= \frac{1}{\frac{2}{3}} - \frac{5}{\frac{5}{6}} = \frac{3}{2} - 6 = -\frac{9}{2}$

*c. $\sum_{n=2}^{\infty}\frac{\cos(n\pi)}{4^n} = \sum_{n=2}^{\infty}\frac{(-1)^n}{4^n}$ $\quad r = \frac{-1}{4}$ $\quad S = \frac{\frac{1}{16}}{1-\left(\frac{-1}{4}\right)} = \frac{1}{16}\cdot\frac{4}{5}$

$= \frac{1}{20}$

*d. $\sum_{n=2}^{\infty}\frac{1}{n(n+1)} = \sum_{n=2}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \frac{1}{2} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \frac{1}{5} + \ldots$

$\frac{A}{n} + \frac{B}{n+1}$ $\quad = \frac{1}{2}$

$A(n+1) + Bn = 1$

13. State the indeterminate form and compute the following limits:

a. $\lim_{n\to\infty}\frac{\ln(n+4)}{n+2}$ $\quad \frac{\infty}{\infty}$ $\quad \overset{H}{=} \lim_{n\to\infty}\frac{\frac{1}{n+4}}{1} = 0$

b. $\lim_{n\to\infty}(3n)^{\frac{2}{n}} = y$ $\quad \infty^0$ $\quad \ln y = \lim_{n\to\infty}\frac{2\ln 3n}{n}$ $\quad \frac{\infty}{\infty}$

$\overset{H}{=} \lim_{n\to\infty}\frac{\frac{2}{n}}{1} = 0$ $\quad \therefore e^0 = 1 = y$

1^∞

c. $\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^{2n} = (e^3)^2 = e^6$

$\frac{0}{0}$

d. $\lim_{x\to 0}\frac{x-\sin(2x)}{x+\sin(2x)} \overset{H}{=} \lim_{x\to 0}\frac{1-2\cos 2x}{1+2\cos 2x} = \frac{-1}{3}$

e. $\lim\limits_{x\to 0}\dfrac{e^{x^2}-1}{2x^2}$ $\frac{0}{0}$ $\overset{L}{=} \lim\limits_{x\to 0} \dfrac{2\cancel{x}e^{x^2}}{4\cancel{x}} = \dfrac{1}{2}$

f. $\lim\limits_{x\to 0+}\left(\dfrac{1}{x}\right)^x = y$ ∞^0

$\ln y = \lim\limits_{x\to 0^+} x\ln\left(\frac{1}{x}\right)$ $0\cdot\infty$

$= \lim\limits_{x\to 0^+} \dfrac{\ln\frac{1}{x}}{\frac{1}{x}} \overset{L}{=} \lim\limits_{x\to 0^+} \dfrac{\frac{-\frac{1}{x^2}}{\frac{1}{x}}}{\frac{-1}{x^2}} = 0$ $\frac{\infty}{\infty}$

$\therefore e^0 = 1 = y$

g. $\lim\limits_{x\to 0}\dfrac{3e^{x/3}-(3+x)}{x^2}$ $\frac{0}{0}$ $\overset{L}{=} \lim\limits_{x\to 0} \dfrac{e^{x/3}-1}{2x}$ $\frac{0}{0}$

$\overset{L}{=} \lim\limits_{x\to 0} \dfrac{\frac{1}{3}e^{x/3}}{2} = \dfrac{1}{6}$

h. $\lim\limits_{x\to\infty}\dfrac{x^2}{\ln x}$ $\frac{\infty}{\infty}$ $\overset{L}{=} \lim\limits_{x\to\infty} \dfrac{2x}{\frac{1}{x}} = \infty$ DNE

i. $\lim\limits_{x\to 0}\dfrac{1+x-e^x}{x(e^x-1)}$ $\frac{0}{0}$ $\overset{L}{=} \lim\limits_{x\to 0} \dfrac{1-e^x}{xe^x+e^x-1}$ $\frac{0}{0}$ $\overset{L}{=} \lim\limits_{x\to 0} \dfrac{-e^x}{xe^x+e^x+e^x} = \dfrac{-1}{2}$

j. $\lim\limits_{x\to 0}\dfrac{\arctan(4x)}{x}$ $\frac{0}{0}$ $\overset{L}{=} \lim\limits_{x\to 0} \dfrac{\frac{4}{1+16x^2}}{1} = 4$

14. Give the derivative of each power series below, and
15. for each of the problems in number 14, give the antiderivative F of the power series so that F(0)=0.

a. $\sum\limits_{n=0}^{\infty}\dfrac{(n+1)x^n}{n^2+2}$

14) $\sum\limits_{n=1}^{\infty} \dfrac{n(n+1)x^{n-1}}{n^2+2}$

15) $\sum\limits_{n=0}^{\infty} \dfrac{\cancel{(n+1)}x^{n+1}}{\cancel{(n+1)}(n^2+2)} + c$ $F(0)=0 \Rightarrow c=0$

$= \sum\limits_{n=0}^{\infty} \dfrac{x^{n+1}}{n^2+2}$

b. $\sum_{n=0}^{\infty} \frac{x^n}{2n+1}$

14) $\sum_{n=1}^{\infty} \frac{n x^{n-1}}{2n+1}$

15) $\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)(2n+1)} + C$ $F(0)=0 \Rightarrow C=0$

16. Evaluate each improper integral and tell why it is improper:

a. $\int_0^{27} x^{-2/3} dx$ $= \lim_{a \to 0^+} \int_a^{27} x^{-2/3} dx$
$= \lim_{a \to 0^+} 3x^{1/3} \Big|_a^{27}$
$= 9 - \lim_{a \to 0^+} 3a^{1/3} = 9$

$\frac{1}{x^{2/3}}$ is discontinuous at $x=0$, which is in $[0, 27]$; OR $x^{-2/3}$ not defined at $x=0$

b. $\int_0^4 \frac{1}{\sqrt{4-x}} dx$ $= \lim_{b \to 4^-} \int_0^b (4-x)^{-1/2} dx$
$= \lim_{b \to 4^-} -2(4-x)^{1/2} \Big|_0^b$
$= \lim_{b \to 4^-} -2(4-b)^{1/2} + 4 = 4$

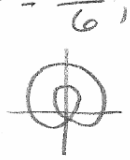
$\frac{1}{\sqrt{4-x}}$ not defined at $x=4$

*c. $\int_{-2}^{0} \frac{1}{x+1} dx$ $= \lim_{c \to -1^-} \int_{-2}^{c} \frac{1}{x+1} dx + \lim_{c \to -1^+} \int_c^0 \frac{1}{x+1} dx$
$= \lim_{c \to -1^-} \ln|x+1| \Big|_{-2}^{c}$
$= -\infty$ $\therefore$ d'ges

$\frac{1}{x+1}$ is discont at $x=-1$, which is in $[-2, 0]$

17. Find the formula for the area of $r = 1 + 2\sin\theta$

$1 + 2\sin\theta = 0$
$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$



a. Inside inner loop

$A = \frac{1}{2} \int_{7\pi/6}^{11\pi/6} (1 + 2\sin\theta)^2 d\theta$

b. Inside outer loop but outside inner loop

$A = 2\left[\frac{1}{2} \int_{-\frac{\pi}{6}}^{\pi/2} (1+2\sin\theta)^2 d\theta - \frac{1}{2} \int_{7\pi/6}^{3\pi/2} (1+2\sin\theta)^2 d\theta\right]$

c. Inside outer loop and below x-axis

$A = 2\left[\frac{1}{2} \int_{-\frac{\pi}{6}}^{0} (1 + 2\sin\theta)^2 d\theta\right.$ OR $2 \cdot \frac{1}{2} \int_{\frac{11\pi}{6}}^{2\pi} (1+2\sin\theta)^2 d\theta$

18. Find the smallest value of n so that the nth degree Taylor Polynomial for $f(x) = ln(1+x)$ centered at x = 0 approximates $ln(2)$ with an error of no more than 0.001. (Also be able to do this with some of the other Taylor Polynomials).

$f(x) = \ln(1+x) \quad f(1) = \ln 2 \quad \Rightarrow x = 1$

$\left|f^{n+1}(x)\right| = \left|\frac{n!}{(1+x)^{n+1}}\right|$

$\le n!$ for $0 \le x \le 1$

let $m = n!$

$|f(1) - P_n(1)| \le \left|\frac{m}{(n+1)!}(x-0)^{n+1}\right|$

$\le \left|\frac{n!}{(n+1)!} \cdot 1^{n+1}\right| \le \frac{1}{1000}$

$\frac{1}{n+1} \le \frac{1}{1000}$

$1000 \le n+1$

$999 \le n$

So $n = 999$.

999^{th} degree Taylor Polynomial

19. Find the radius of convergence and interval of convergence for the following Power series:

a. $\sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{(n+1)3^{n+1}}$

$\lim_{n\to\infty} \left|\frac{(x-2)^{n+2}}{(n+2)3^{n+2}} \cdot \frac{(n+1)3^{n+1}}{(x-2)^{n+1}}\right| = \frac{|x-2|}{3} \cdot 1 < 1$

$|x-2| < 3 \quad \boxed{R=3}$

$-3 < x-2 < 3$

$-1 < x < 5$

at $x=-1$ c'ges

at $x=5$ d'ges

$\boxed{[-1,5)}$

b. $\sum_{n=0}^{\infty} \frac{1}{3^n}(x-1)^n$

$\left(\frac{x-1}{3}\right)^n$

geometric

$\left|\frac{x-1}{3}\right| < 1$

$|x-1| < 3$

$-3 < x-1 < 3$

$-2 < x < 4$

$\boxed{R=3}$ $\boxed{(-2,4)}$

c. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{4^n}$

$\lim_{n\to\infty} \left|\frac{x^{n+1}}{4^{n+1}} \cdot \frac{4^n}{x^n}\right| = \frac{|x|}{4} < 1$

$|x| < 4$

$-4 < x < 4$ $\boxed{R=4}$

$\boxed{(-4,4)}$

at $x=-4$ d'ges

at $x=4$ d'ges

d. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n n!}{n^n}$ $\lim_{n\to\infty} \left| \frac{x^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{x^n n!} \right| = \lim_{n\to\infty} |x| \left(\frac{n}{n+1}\right)^n$

$= |x| e^{-1} < 1$

$|x| < e$ $\boxed{R=e}$

$-e < x < e$

at $x=e$ c'ges

at $x=-e$ d'ges

$(-e, e]$

*e. $\sum_{n=1}^{\infty} \frac{x^n}{2^n}$ $\left|\frac{x}{2}\right| < 1$ $|x| < 2$ $R=2$

$-2 < x < 2$ $(-2, 2)$

geometric

20. Use logarithmic differentiation to find the derivative of:

a. $y = (3x-1)^{\sin(x)}$

$\ln y = (\sin x)\ln(3x-1)$

$\frac{y'}{y} = \frac{3(\sin x)}{3x-1} + (\cos x)\ln(3x-1)$

$y' = (3x-1)^{\sin x}\left[\frac{3\sin x}{3x-1} + (\cos x)\ln(3x-1)\right]$

b. $y = (x+1)^{\ln(x)}$

$\ln y = (\ln x)\ln(x+1)$

$\frac{y'}{y} = \frac{\ln x}{x+1} + \frac{\ln(x+1)}{x}$

$y' = (x+1)^{\ln x}\left[\frac{\ln x}{x+1} + \frac{\ln(x+1)}{x}\right]$

c. $y = (x^2+2)^{\left(\frac{1}{\ln x}\right)}$

$\ln y = \frac{1}{\ln x}\ln(x^2+2)$

$\frac{y'}{y} = \frac{1}{\ln x} \cdot \frac{2x}{x^2+2} + \ln(x^2+2) \cdot \frac{(-1)}{x(\ln x)^2}$

$y' = (x^2+2)^{1/\ln x}$[above]

21. Determine the convergence or divergence for each series with the given general term:

Series	Converge or Diverge?	Test used (there may be others that work)
$\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}}$	D	P-series
$\sum_{n=1}^{\infty} \frac{2^n}{n^3}$	D	Ratio
$\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n}\right)$	C	Telescoping or comparison

$\sum_{n=1}^{\infty} \frac{3^{2n}}{n!}$	C	Ratio
$\sum_{n=1}^{\infty} \cos(\pi n)$	D	Basic Div.
$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n}$	D	P-series
$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{3n^3+1}$	C	A.S.T
$\sum_{n=0}^{\infty} 3\left(-\frac{1}{2}\right)^n$	C	geometric
$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$	C	integral

$\sum_{n=1}^{\infty} ne^{-n^3}$	C	Ratio
$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^n$	D	Basic Div.
$\sum_{n=1}^{\infty} \frac{1}{n^3+1}$	C	BCT, LCT, $\frac{1}{n^3}$
$\sum_{n=0}^{\infty} \left(\frac{2}{9}\right)^n$	C	geometric
$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$	C	Ratio
$\sum_{n=2}^{\infty} \frac{10n^2+n-2}{2n^6+7n-1}$	C	LCT $\frac{1}{n^4}$

$\sum_{n=1}^{\infty} \frac{n^2+3n-2}{\sqrt{4n^9+n-1}}$	C	compare $\frac{1}{n^{5/2}}$

(*) from here to the end.

22. A culture of bacteria is growing in such a way that the number of bacteria is changing at a rate proportional to the number of bacteria. If there are initially 10,000 bacteria, and 12,000 bacteria are present six hours later, what is the doubling time for the culture? (give your answer in terms of *ln*). $C=10000$

$12\cancel{000} = 10\cancel{000}e^{6k}$

$\frac{6}{5} = e^{6k}$

$\ln 6/5 = 6k$

$K = \frac{\ln 6/5}{6}$

$DT = \ln 2 / \left(\frac{\ln 6/5}{6}\right)$

$= \frac{6\ln 2}{\ln 6/5}$

23. Give the solution to $\frac{dy}{dx} = -3y,\ y(0) = -2.$

$\int \frac{dy}{y} = \int -3dx$

$\ln|y| = -3x + c$

$y = Ce^{-3x}$

$-2 = C$

$\boxed{y = -2e^{-3x}}$

24. Identify the geometric shape given by the parameterization $x(t) = -2 + 3\cos(t), y(t) = 1 + 3\sin(t)$

$\frac{x+2}{3} = \cos t \quad \frac{y-1}{3} = \sin t$

$\left(\frac{x+2}{3}\right)^2 + \left(\frac{y-1}{3}\right)^2 = 1$ circle; R=3

25. Give a parameterization for the line segment from the point (1, 6) to the point (–3, 1).

$x(t) = 1 + (-3-1)t = 1 - 4t$ $\quad 0 \le t \le 1$

$y(t) = 6 + (1-6)t = 6 - 5t$

26. Give a parameterization for the curve given in polar coordinates by $r = 1 + \sin(\theta)$. $\quad x = r\cos\theta \quad y = r\sin\theta$

$r^2 = r + r\sin\theta$

$x^2 + y^2 = r + y$

$x^2 + y^2 = \sqrt{x^2+y^2} + y$

27. Give the formula for the arc length of a curve parameterized by $x(t)=\cos(t)$, $y(t)=t^2$ for $0 \le t \le 1$.

$x'(t) = -\sin t$

$y'(t) = 2t$

$L = \int_0^1 \sqrt{(-\sin t)^2 + (2t)^2}\, dt$

28. Write the line $y = x$ in polar coordinates.

$\frac{y}{x} = 1 \quad \tan\theta = \frac{y}{x} = 1 \quad \boxed{\theta = \frac{\pi}{4}}$

29. Use long division to rewrite $\dfrac{x^4}{x^3+x^2+1}$ $= x - 1 + \dfrac{x^2 - x - 1}{x^3+x^2+1}$

$$\begin{array}{r} x - 1 \\ x^3+x^2+1 \overline{)\, x^4 \qquad\qquad} \\ \underline{x^4 + x^3 + x} \\ -x^3 - x \\ \underline{-x^3 - x^2 - 1} \\ x^2 - x - 1 \end{array}$$

ugly

30. Give the partial fraction decomposition for $\dfrac{2x+1}{(x-1)^2(x^2+1)}$

$\frac{2x+1}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1} = \frac{-\frac{1}{2}}{x-1} + \frac{3/2}{(x-1)^2} + \frac{\frac{1}{2}x - 1}{x^2+1}$

$x=1 \Rightarrow \boxed{B = 3/2}$

$2x+1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$

$= A(x^3+x-x^2-1) + Bx^2 + B + Cx^3 - 2Cx^2 + Cx + Dx^2 - 2Dx + D$

$x^3 \quad 0 = A + C \quad \boxed{A = -1/2}$

$x^2 \quad 0 = -A + B - 2C + D \quad 1 = 2C \quad \boxed{C = 1/2}$

$x=0 \quad 1 = -A + B + D$

$D = -1$

31. Give the greatest lower bound of the set $\{\, x \mid x^2 + 3x - 10 < 0 \,\}$.

$(x+5)(x-2) < 0$

$x = -5 \quad x = 2$

$(-5, 2)$

GLB $= -5$

LUB $= 2$

32. Suppose $f(x) = \sum_{n=0}^{\infty} \dfrac{x^{2n-1}}{(2n)!}$. Give the 13^{th} derivative of f at $x = 0$.

$a_k x^k = \frac{f^{13}(0)}{13!} x^{13}$

$2n - 1 = 13$

$n = 7$

$\frac{1}{14!} = \frac{f^{13}(0)}{13!}$

$f^{13}(0) = \frac{1}{14}$

$f^k(0) = k!\, a_k$

any Taylor Series centered at $x=0$

33. Give the 5th degree Taylor polynomial for e^x centered at 0.

$$P_5(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

34. Give the 6th degree Taylor polynomial for $cos(x)$ centered at 0.

$$P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

35. Give the Taylor series expansion for $f(x) = e^{-x}$ centered at 0.

$$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \dots$$

$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

36. Give a power series expansion for $f(x) = ln(x)$ centered at 1.

$$\frac{d}{dx} \ln x = \frac{1}{x} = \frac{1}{1+(x-1)} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

$r = x-1$

$|x-1| < 1$

$-1 < x-1 < 1$

$0 < x < 2$

$$\ln x = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1} \cancel{+c}$$

$\ln 1 = 0 \Rightarrow c = 0$

$R = 1 \quad (0, 2]$

37. Give a power series expansion for $f(x) = sin(3x)$ centered at 0.

$$f(x) = 3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k (3x)^{2k+1}}{(2k+1)!}$$

38. Give a power series expansion for $f(x) = \dfrac{1}{(1+x)^2}$ centered at 0.

$f(x) = (1+x)^{-2}$

$f'(x) = -2(1+x)^{-3}$

$f''(x) = 6(1+x)^{-4}$

$f'''(x) = -24(1+x)^{-5}$

$$f(x) = 1 + \frac{-2}{1!}x + \frac{6}{2!}x^2 - \frac{24}{3!}x^3 + \dots$$

$$= 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$$

39. $f(1) = -1, f'(1) = 2, f''(1) = -1$. Give the 2nd degree Taylor polynomial for f centered at 1.

$$P_2(x) = -1 + \frac{2}{1!}(x-1)^1 + \frac{-1}{2!}(x-1)^2$$

$$= -1 + 2(x-1) - \frac{(x-1)^2}{2}$$

40. Rewrite $f(x) = x^3 + 2x^2 - x + 1$ in powers of $(x+1)$.

at -1 $\quad f(-1) = 3$

$f'(x) = 3x^2 + 4x - 1$

$f''(x) = 6x + 4$

$f'''(x) = 6$

$$f(x) = 3 - 2(x+1) - \frac{2(x+1)^2}{2!} + \frac{6(x+1)^3}{3!}$$

$$= 3 - 2(x+1) - (x+1)^2 + (x+1)^3$$

41. Give a power series representation for $arctan(2x)$ and give the radius of convergence.

$$\frac{d}{dx} \arctan 2x = \frac{2}{1+4x^2} = 2\left(\frac{1}{1-(-4x^2)}\right) \qquad r = -4x^2$$

$\arctan 0 = 0$

$$2\sum_{n=0}^{\infty} (-4x^2)^n = 2\sum_{n=0}^{\infty} (-4)^n x^{2n}$$

$$\arctan 2x = 2\sum \frac{(-4)^n x^{2n+1}}{2n+1} + \cancel{C}^{0}$$

$\boxed{42.}$ Give a value of n so that the Taylor polynomial of degree n for $f(x) = sin(x)$ centered at 0 can be used to approximate $f(x)$ within 10^{-4} on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

largest x is $\frac{1}{2}$

$$|f(x) - P_n(x)| = \left|\frac{f^{k+1}(c)}{(k+1)!} x^{k+1}\right|$$

$$\leq \frac{1}{(k+1)!}\left(\frac{1}{2}\right)^{k+1} < \frac{1}{10000}$$

$$\frac{1}{2^{k+1}(k+1)!} < \frac{1}{10000}$$

$$10000 < 2^{k+1}(k+1)!$$

$k=5 \qquad 46080$

$k=4 \qquad 3840$

$\boxed{\therefore n = 5}$