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MATH 1432 **ANSWERS: EXAM [redacted] REVIEW**

L'Hôpital's Rule:

(a) indeterminate; $\frac{0}{0}$; $-\frac{1}{2}$

(b) not indeterminate; $\frac{1}{2}$

(c) indeterminate; $0 \times \infty$; -1

(d) indeterminate; 1^{∞}; e^4

(e) indeterminate; $\frac{0}{0}$; $\frac{1}{2}$

(f) indeterminate; $\infty - \infty$; does not exist

Improper Integrals:

(a) improper (integrand unbounded); converges; 4

(b) not improper

(c) improper (infinite interval); converges; 1

(d) improper (infinite interval); converges; π

(e) improper (infinite interval; integrand unbounded); diverges

Infinite Series, General:

(a) $a_k = (-1)^k\, 4\left(\frac{3}{4}\right)^k$; $\sum_{k=0}^{\infty}(-1)^k\, 4\left(\frac{3}{4}\right)^k = \sum_{k=0}^{\infty}(-1)^k \frac{3^k}{4^{k-1}}$; $\frac{16}{7}$

(b) $\sin(\pi/2) + \sin(3\pi/2) + \sin(5\pi/2) + \cdots = 1 - 1 + 1 - 1 + \cdots = \sum_{k=0}^{\infty} \sin[(2k+1)\pi/2]$;

$s_n = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd}; \end{cases}$ the series diverges.

(c) $a_k = (-1)^k \frac{k+2}{k+1}$, $k = 0, 1, 2, \ldots$; diverges, a_k does not have limit 0.

Nonnegative Series:

(a) converges; ratio test

(b) diverges; ratio test or root test

(c) diverges; integral test

(d) converges; limit comparison test — $\sum \frac{1}{k^{3/2}}$

Arbitrary Series

(a) (*i*) conditionally convergent; $\frac{1}{\sqrt{k^2+3k+2}} \approx \frac{1}{k}$ for large k.

(*ii*) absolutely convergent; $\frac{|\sin(k)|}{(k+1)^2} \le \frac{1}{(k+1)^2}$

(*iii*) absolutely convergent; $\sum_{k=0}^{\infty} \frac{k^2}{2^k}$ converges — ratio test or root test.

(b) (*i*) diverges; as $k \to \infty$, $[a_k]^{1/k} = \left[\dfrac{k^2}{(\ln 2)^k}\right]^{1/k} \to \dfrac{1}{\ln 2} \cong \dfrac{1}{0.69} > 1$

(*ii*) converges; as $k \to \infty$, $[a_k]^{1/k} = \left[\dfrac{k^2}{(\ln 3)^k}\right]^{1/k} \to \dfrac{1}{\ln 3} \cong \dfrac{1}{1.1} < 1$

Power Series

(a) $\sum \dfrac{(-1)^k}{4^k \ln k}\, x^k : \quad R = 4,\ (4, 4]$

(b) $\sum \dfrac{1}{k^3+1}\, x^k : \quad R = 1,\ [-1, 1]$

(c) $\sum \dfrac{1}{(k+1)3^k}\,(x+1)^k : \quad R = 3,\ [-4, 2)$

(d) $\sum \dfrac{(-2)^k}{\sqrt{k+1}}\, x^k : \quad R = 1/2,\ (-1/2, 1/2]$

(e) $\sum \dfrac{k!}{4^k}\,(x-3)^k : \quad R = 0,\ \{3\}$

(f) $\sum \dfrac{k}{k^3+2}\, x^k : \quad R = 1,\ [-1, 1]$

Taylor Polynomials, Taylor Series

(a) $P_4(x) = 1 + 3x + \dfrac{3}{2}\,x^2 - \dfrac{1}{2}\,x^3 + \dfrac{3}{8}\,x^4$

(b) $\cosh x = 1 + \dfrac{1}{2!}x^2 + \dfrac{1}{4!}x^4 + \cdots = \displaystyle\sum_{k=0}^{\infty} \dfrac{1}{2k!}x^{2k}.$

(c) $P_5(x) = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}(x-\pi/6) - \dfrac{1}{4}(x-\pi/6)^2 - \dfrac{\sqrt{3}}{12}(x-\pi/6)^3 + \dfrac{1}{48}(x-\pi/6)^4 + \dfrac{\sqrt{3}}{240}(x-\pi/6)^5.$

(d) $f^{(9)}(0) = 2^8\, 9!$

Approximation by Taylor Polynomials

(a) (*i*) $\text{error} = |f(\tfrac{1}{2}) - P_4(\tfrac{1}{2})| \le \dfrac{\max|f^{(5)}(t)|}{5!}\left(\dfrac{1}{2}\right)^5 \le \dfrac{2}{5!}\left(\dfrac{1}{2}\right)^5 = \dfrac{1}{(120)(16)} = \dfrac{1}{1920} \cong 0.00052$

(*ii*) $\text{error} = |f(\tfrac{1}{2}) - P_n(\tfrac{1}{2})| \le \dfrac{\max|f^{(n+1)}(t)|}{(n+1)!}\left(\dfrac{1}{2}\right)^{n+1} \le \dfrac{2}{(n+1)!}\cdot\dfrac{1}{2^{n+1}} < 0.00005$

$\implies 2^n(n+1)! > 20{,}000 \implies n \ge 5.$ Therefore, take $n = 5$.

(b) (*i*) $\text{error} = |e^{1/2} - P_4(\tfrac{1}{2})| \le \dfrac{\max|e^t|}{5!}\left(\dfrac{1}{2}\right)^5 < \dfrac{3}{5!}\left(\dfrac{1}{2}\right)^5 = \dfrac{3}{(120)(32)} \cong 0.00078 \quad (\max|e^t| < 3)$

(*ii*) $|e^{1/2} - P_n(\tfrac{1}{2})| \le \dfrac{\max|e^t|}{(n+1)!}\left(\dfrac{1}{2}\right)^{n+1} \le \dfrac{3}{(n+1)!}\cdot\dfrac{1}{2^{n+1}} < \dfrac{1}{10{,}000}$

$\implies 2^{n+1}(n+1)! > 30{,}000 \implies n \ge 5.$ Therefore, take $n = 5$.

(c) $\cos x = \dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}(x-\pi/4) - \dfrac{\sqrt{2}}{4}(x-\pi/4)^2 + -\dfrac{\sqrt{2}}{12}(x-\pi/4)^3 + \cdots.$

$\cos 40^\circ = \cos(45^\circ - 5^\circ) = \cos(\pi/4 - \pi/36)$

(*i*) error $= |\cos(\pi/4 - \pi/36) - P_2(\pi/4 - \pi/36)| \le \dfrac{\max|\sin t|}{3!} \left(\dfrac{\pi}{36}\right)^3 \le \dfrac{1}{3!}\left(\dfrac{\pi}{36}\right)^3 \cong 0.00011$

(*ii*) $|\cos(\pi/4 - \pi/36) - P_n(\pi/4 - \pi/36)| \le \dfrac{1}{(n+1)!}\left(\dfrac{\pi}{36}\right)^{n+1} = \dfrac{\pi^{n+1}}{(36)^{n+1}(n+1)!} < 0.00005$

$\implies n \ge 3.$