L'Hopital

a) $\lim_{x\to 0} \frac{1+x-e^x}{x^2} \overset{0/0}{=} \lim_{x\to 0} \frac{1-e^x}{2x} \overset{0/0}{=} \lim_{x\to 0} \frac{-e^x}{2} = \frac{-1}{2}$

b) $\lim_{x\to 1} \frac{x+\ln x}{2x^2} = \frac{1}{2}$

c) $\lim_{x\to \pi/2} \underset{}{\overset{0\cdot\infty}{(x-\pi/2)\tan x}} = \lim_{x\to \frac{\pi}{2}} \frac{(x-\frac{\pi}{2})\sin x}{\cos x} \; \frac{0}{0}$

$= \lim_{x\to \frac{\pi}{2}} \frac{(x-\frac{\pi}{2})\cos x+\sin x}{-\sin x} = -1$

d) $\lim_{x\to \infty} \underset{1^\infty}{\left(1+\frac{2}{x}\right)^{2x}} = \lim_{x\to \infty} \left[\left(1+\frac{2}{x}\right)^x\right]^2 = \left(e^2\right)^2 = e^4$

e) $\lim_{x\to 0} \frac{1-\cos x}{x^2} \overset{0/0}{=} \lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2}$

f) $\lim_{x\to 0^+} \underset{\infty-\infty}{\left(\frac{1}{x}-\frac{1}{\sqrt{x}}\right)} = \lim_{x\to 0^+} \frac{\sqrt{x}-x}{x\sqrt{x}} \overset{0/0}{=} \lim_{x\to 0^+} \frac{\frac{1}{2\sqrt{x}}-1}{\frac{3}{2}x^{1/2}}$

$= \lim_{x\to 0^+} \frac{1-2\sqrt{x}}{3x} \Rightarrow \infty \Rightarrow \text{DNE}$

Improper Integrals

a) $\int_2^6 \frac{1}{\sqrt{x-2}}\,dx = \lim_{a\to 2^+} \int_a^6 (x-2)^{-1/2}\,dx$

improper
$x=2$ unbounded $= \lim_{a\to 2^+} 2(x-2)^{1/2}\Big|_a^6$

$= \lim_{a\to 2^+} \left[2(2) - 2(a-2)^{1/2}\right]$

$= 4$ $\therefore$ converges

b) $\int_{-1}^1 \frac{1}{4-x^2}\,dx = \arcsin\frac{x}{2}\Big|_{-1}^1$

not improper

$= \arcsin\frac{1}{2} - \arcsin\frac{-1}{2}$

$= \frac{\pi}{6} - \left(\frac{-\pi}{6}\right) = \frac{\pi}{3}$

c) $\int_0^\infty xe^{x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx$

improper;
infinite

$= \lim_{b\to\infty} \left(-xe^{-x} - e^{-x}\right)\Big|_0^b$

$$\begin{array}{ccc} x & e^{-x} & + \\ 1 & -e^{-x} & - \\ & e^{-x} & + \end{array}$$

$= \lim_{b\to\infty}\left(\frac{-b}{e^b} - \frac{1}{e^b} - (0-1)\right)$

$= 0-0+1$

$= 1$ $\therefore$ converges

d) $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

improper;
infinite

$$= \lim_{a\to-\infty} \int_a^0 \frac{1}{1+x^2}\,dx + \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}\,dx$$

$$= \lim_{a\to\infty} (\arctan x)\Big|_a^0 + \lim_{b\to\infty} (\arctan x)\Big|_0^b$$

$$= \lim_{a\to\infty} (\arctan 0 - \arctan a) + \lim_{b\to\infty} (\arctan b - \arctan 0)$$

$$= 0 - \left(-\frac{\pi}{2}\right) \quad + \frac{\pi}{2} - 0$$

$$= \pi \qquad \therefore \text{ converges}$$

e) $$\int_1^{\infty} \frac{1}{\sqrt{x-1}}\,dx = \int_1^{5} \frac{1}{\sqrt{x-1}}\,dx + \int_5^{\infty} \frac{1}{\sqrt{x-1}}$$

$\leftarrow$ any # > 1

improper;
∞ at $x=1$

$$= \lim_{a\to1^+} \int_a^5 (x-1)^{-1/2}\,dx + \lim_{b\to\infty} \int_5^b (x-1)^{-1/2}\,dx$$

$$= \lim_{a\to1^+} 2(x-1)^{1/2}\Big|_a^5 + \lim_{b\to\infty} 2(x-1)^{1/2}\Big|_5^b$$

$$= \lim_{a\to1^+} 2(2) - 2(a-1)^{1/2} + \lim_{b\to\infty} \left(2(b-1)^{1/2} - 2(2)\right)$$

$$= \quad 4 \quad + \infty$$

$\therefore$ diverges

Infinite Series, General

a) $4 - 3 + \frac{9}{4} - \frac{27}{16} + \cdots$ $\quad r = \frac{-3}{4}$ $\quad \frac{a_{n+1}}{a_n} = r$

$$\sum_{k=0}^{\infty} 4\left(-\frac{3}{4}\right)^k = \frac{4}{1-(-3/4)} \qquad |r| = \left|\frac{-3}{4}\right| < 1 \therefore \text{ converges}$$

$$= \frac{4 \cdot 4}{7} = \frac{16}{7}$$

b) $\sin\frac{\pi}{2} + \sin\frac{3\pi}{2} + \sin\frac{5\pi}{2} + \cdots = \sum_{k=0}^{\infty} \sin\left[(2k+1)\pi/2\right].$

$= 1 - 1 + 1 - 1 + \cdots$

$S_0 = 1$

$S_1 = 1 - 1 = 0$ $\qquad S_n = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

$S_2 = 1 - 1 + 1 = 1$

$S_3 = 1 - 1 + 1 - 1 = 0$ $\qquad 1, 0, 1, 0, \ldots$ diverges

$\vdots$

c) $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots$ $\qquad a_k = \frac{(-1)^{k+1}(k+1)}{k}$ $\quad k = 1, 2, 3, \ldots$

1 2 3 4

diverges; $\lim_{k\to\infty} a_k \neq 0$

Nonnegative Series

a) $\sum_{K=0}^{\infty} \frac{K^2 2^K}{(K+1)!}$ $\quad \lim_{K\to\infty} \left| \frac{(K+1)^2 2^{K+1}}{(K+2)!} \cdot \frac{(K+1)!}{K^2 2^K} \right|$

$$= \lim_{K\to\infty} \left| \underbrace{\frac{(K+1)^2}{K^2}}_{\to 1} \; \underbrace{2}_{\to 2} \cdot \underbrace{\frac{2^K}{2^K}}_{\to 1} \; \underbrace{\frac{1}{(K+2)}}_{\to 0} \right| = 0 < 1$$

$\therefore$ converges; Ratio test

b) $\sum_{K=0}^{\infty} \frac{3^{K+1}}{(K+1)^2 e^K}$ $\quad \lim_{K\to\infty} \left| \frac{3^{K+2}}{(K+2)^2 e^{K+1}} \cdot \frac{(K+1)^2 e^K}{3^{K+1}} \right|$

$$= \lim_{K\to\infty} \left| \frac{9 \cdot \cancel{3^K} \cdot (K+1)^2}{3 \cdot \cancel{3^K} \underbrace{(K+2)^2}_{\to 1} e} \right|$$

$= \frac{3}{e} > 1 \quad \therefore$ diverges Ratio test

c) $\sum_{K=1}^{\infty} \frac{\ln K}{K}$ $\quad \int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x}\,dx$

$f(x) = \frac{\ln x}{x}$

cont; pos; dec

$$= \lim_{b\to\infty} \left. \frac{(\ln x)^2}{2} \right|_1^b$$

$$= \lim_{b\to\infty} \frac{(\ln b)^2}{2} - 0$$

$= \infty \;\therefore$ diverges

$\therefore$ diverges; integral test

d) $\sum_{k=0}^{\infty} \frac{2k+1}{\sqrt{k^5+3k^3+4}}$ compare to $\frac{1}{k^{3/2}}$; c'ges $p=3/2$

$\lim_{k\to\infty} \frac{2k+1}{\sqrt{k^5+3k^3+4}} \cdot \frac{k^{3/2}}{1} = \lim_{k\to\infty} \frac{2k^{5/2}+k^{3/2}}{\sqrt{k^5+3k^3+4}} = 2$ f + p

$\therefore$ c'ges limit comparison

Arbitrary Series

a) i) $\sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{k^2+3k+2}}$ $\sum \frac{1}{\sqrt{k^2+3k+2}}$ d'ges $\left(\frac{1}{k}\right)$

$\therefore$ conditionally c'gt

$\left.\begin{array}{l} \frac{1}{\sqrt{(k+1)^2+3(k+1)+2}} < \frac{1}{\sqrt{k^2+3k+2}} \\ \lim_{k\to\infty} \frac{1}{\sqrt{k^2+3k+2}} = 0 \end{array}\right\}$ c'ges alt. ser.

ii) $\sum_{k=0}^{\infty} \frac{\sin k}{(k+1)^2}$ $\sum_{k=0}^{\infty} \frac{|\sin k|}{(k+1)^2}$ compare to $\frac{1}{k^2}$ c'ges $p=2$

$\frac{|\sin k|}{(k+1)^2} \le \frac{1}{(k+1)^2} < \frac{1}{k^2}$ Basic comp

$k \neq 0$

$\therefore$ c'ges absolutely

iii) $\sum_{k=0}^{\infty} \frac{(-1)^k k^2}{2^k}$ $\sum \left| \frac{(-1)^k k^2}{2^k} \right|$ $\lim_{k\to\infty} \sqrt[k]{\left|\frac{k^2}{2^k}\right|} = \frac{1}{2} < 1$

$\therefore$ c'ges, Root

$\therefore$ c'ges absolutely

b) $\sum_{k=0}^{\infty} \frac{k^2}{(\ln 2)^k}$ d'ges $\sum_{k=0}^{\infty} \frac{k^3}{(\ln 3)^k}$ c'ges exponential trumps power

$\ln 2 < 1$ $\ln 3 > 1$

(Root test works on both)

Power Series

a) $\sum_{k=2}^{\infty} \frac{(-1)^k}{4^k \ln k} x^k$ $\lim_{k\to\infty} \left| \frac{x^{k+1}}{4^{k+1}\ln(k+1)} \cdot \frac{(\ln k)\, 4^k}{x^k} \right|$

$= |x| \lim_{k\to\infty} \left| \frac{4^k \ln k}{4 \cdot 4^k \ln(k+1)} \right|$

$= |x| \cdot \frac{1}{4} < 1$

$|x| < 4$

$-4 < x < 4$

$\boxed{R=4 \quad (-4,4]}$

at $x=-4$

$\sum \frac{(-1)^k (-4)^k}{4^k \ln k}$

$= \sum \frac{1}{\ln k}$ d'ges

at $x=4$

$\sum \frac{(-1)^k 4^k}{4^k \ln k}$

$= \sum \frac{(-1)^k}{\ln k}$ c'ges

b) $\sum_{k=0}^{\infty} \frac{1}{k^3+1} x^k$ $\lim_{k\to\infty} \left| \frac{x^{k+1}}{(k+1)^3+1} \cdot \frac{k^3+1}{x^k} \right|$

$= |x| \lim_{k\to\infty} \left| \frac{k^3+1}{(k+1)^3+1} \right|$

$= |x| < 1$

$-1 < x < 1$

$\boxed{R=1 \quad [-1,1]}$

at $x=-1$

$\sum_{k=0}^{\infty} \frac{(-1)^k}{k^3+1}$

c'ges

at $x=1$

$\sum_{k=0}^{\infty} \frac{1}{k^3+1}$

c'ges

c) $\sum_{K=0}^{\infty} \frac{1}{(K+1)3^K}(x+1)^K$ $\lim_{K\to\infty}\left|\frac{(x+1)^{K+1}}{(K+2)3^{K+1}}\cdot\frac{(K+1)3^K}{(x+1)^K}\right|$

$= |x+1| \lim_{K\to\infty}\left|\frac{3^K\,(K+1)}{3\cdot 3^K\,(K+2)}\right|$

$= |x+1|\cdot\frac{1}{3} < 1$

$|x+1| < 3$

$-3 < x+1 < 3$

$-4 < x < 2$

$\boxed{R=3 \quad [-4, 2)}$

at $x=-4$

$\sum \frac{1}{(K+1)3^K}(-3)^K$

c'ges

at $x=2$

$\sum \frac{3^K}{(K+1)3^K}$ d'ges

d) $\sum_{K=0}^{\infty} \frac{(-2)^K}{\sqrt{K+1}} x^K$ $\lim_{K\to\infty}\left|\frac{(-2)^{K+1}x^{K+1}}{\sqrt{K+2}}\cdot\frac{\sqrt{K+1}}{(-2)^K x^K}\right|$

$= |x| \lim_{K\to\infty}\left|\frac{-2\cdot\sqrt{K+1}}{\sqrt{K+2}}\right|$

$= |x|\cdot 2 < 1$

$|x| < \frac{1}{2}$

$-\frac{1}{2} < x < \frac{1}{2}$

$\boxed{R=\frac{1}{2} \quad (-\frac{1}{2}, \frac{1}{2}]}$

at $x=-\frac{1}{2}$

$\sum \frac{(-2)^K(-\frac{1}{2})^K}{\sqrt{K+1}}$

$\frac{1}{\sqrt{K+1}}$ d'ges

at $x=1/2$

$\sum \frac{(-2)^K(\frac{1}{2})^K}{\sqrt{K+1}}$ c'ges

$\frac{(-1)^K}{\sqrt{K+1}}$

e) $\sum_{K=0}^{\infty} \frac{K!}{4^K}(x-3)^K \quad \lim_{K\to\infty}\left|\frac{(K+1)!\,(x-3)^{K+1}}{4^{K+1}}\cdot\frac{4^K}{K!\,(x-3)^K}\right|$

$= |x-3| \lim_{K\to\infty}\left|\frac{(K+1)}{4}\right|$

$= |x-3|\cdot\infty < 1$

$x=3$

$\boxed{R=0 \quad \{3\}}$

f) $\sum_{K=0}^{\infty} \frac{K}{K^3+2} x^K \quad \lim_{K\to\infty}\left|\frac{(K+1)\,x^{K+1}}{(K+1)^3+2}\cdot\frac{K^3+2}{K\,x^K}\right|$

$= |x| \lim_{K\to\infty}\left|\frac{K+1}{K}\cdot\frac{K^3+2}{(K+1)^3+2}\right|$

$= |x| < 1$

$-1 < x < 1$

$\boxed{R=1 \quad [-1,1]}$

at $x=-1$

$\sum \frac{K(-1)^K}{K^3+2}$ c'ges

at $x=1$

$\sum \frac{K}{K^3+2}$ c'ges

Taylor Polynomials, Taylor Series

a) $f(x) = (1+2x)^{3/2}$ $\qquad f(0) = 1$

$f'(x) = \frac{3}{2}(1+2x)^{1/2} \cdot 2 = 3(1+2x)^{1/2}$ $\qquad f'(0) = 3$

$f''(x) = \frac{3}{2}(1+2x)^{-1/2} \cdot 2 = 3(1+2x)^{-1/2}$ $\qquad f''(0) = 3$

$f'''(x) = \frac{-3}{2}(1+2x)^{-3/2} \cdot 2 = -3(1+2x)^{-3/2}$ $\qquad f'''(0) = -3$

$f^{(4)}(x) = \frac{9}{2}(1+2x)^{-5/2} \cdot 2 = 9(1+2x)^{-5/2}$ $\qquad f^{(4)}(0) = 9$

$$P_4(x) = 1 + 3x + \frac{3}{2}x^2 - \frac{3}{3!}x^3 + \frac{9}{4!}x^4 \qquad \frac{\cancel{9}\,3}{4\cdot\cancel{3}\cdot 2}$$

$$= 1 + 3x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \frac{3}{8}x^4$$

b) $f(x) = e^x$ $\qquad g(x) = \cosh x = \frac{e^x + e^{-x}}{2}$

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$$\cosh x = \frac{1}{2}\left[1 + \cancel{x} + \frac{x^2}{2!} + \cancel{\frac{x^3}{3!}} + \dots\right] + \frac{1}{2}\left[1 - \cancel{x} + \frac{x^2}{2!} - \cancel{\frac{x^3}{3!}} + \dots\right]$$

$$= \frac{1}{2}\left[2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \dots\right]$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$= \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k}$$

c) $f(x) = \sin x$ $\qquad f(\frac{\pi}{6}) = \frac{1}{2}$

$f'(x) = \cos x$ $\qquad f'(\frac{\pi}{6}) = \sqrt{3}/2$

$f''(x) = -\sin x$ $\qquad f''(\frac{\pi}{6}) = -1/2$

$f'''(x) = -\cos x$ $\qquad f'''(\frac{\pi}{6}) = -\sqrt{3}/2$

$f^{(4)}(x) = \sin x$ $\qquad f^{(4)}(\frac{\pi}{6}) = 1/2$

$f^{(5)}(x) = \cos x$ $\qquad f^{(5)}(\frac{\pi}{6}) = \sqrt{3}/2$

$$P_5(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \tfrac{\pi}{6})^1 - \frac{1}{2\cdot 2!}(x - \tfrac{\pi}{6})^2 - \frac{\sqrt{3}}{2\cdot 3!}(x - \tfrac{\pi}{6})^3$$

$$+ \frac{1}{2\cdot 4!}(x - \tfrac{\pi}{6})^4 + \frac{\sqrt{3}}{2\cdot 5!}(x - \tfrac{\pi}{6})^5$$

d) $f(x) = \frac{x}{1+2x} = x - 2x^2 + 4x^3 - 8x^4 + \dots$

$$= \sum (-1)^{k-1} 2^{k-1} x^k$$

$\frac{f^{(9)}(0)}{9!} x^9 = a_9 x^9$ $\quad k=9 \quad a_9 = (-1)^8 2^8 = 2^8$

$$\frac{f^{(9)}(0)}{9!} = 2^8$$

$$f^{(9)}(0) = 2^8 \cdot 9!$$

Approximation by Tayloe Polynomials

a) $P_4(0.5)$ $\quad f(0.5)$ $\quad |f^n(x)| \leq 2$

i) $|R_4| \leq \left| \frac{2}{5!} \left(\frac{1}{2}\right)^5 \right| = \left| \frac{1}{16 \cdot 5!} \right| = \frac{1}{1920}$

ii) correct to 4 decimal points $< .00005$

$$\left| \frac{2}{(n+1)!} \left(\frac{1}{2}\right)^{n+1} \right| < .00005$$

↑ Rounds up to .0001 hence less than only

$$\frac{1}{2^n (n+1)!} < \frac{5}{100000}$$

20000

$$\frac{1}{2^n (n+1)!} < \frac{1}{10000}$$

	$2^n (n+1)!$
$n=4$	$16 \cdot 5! = 1920$
$n=5$	$32 \cdot 6! = 23040$

$\therefore n=5$ works

b) $f(x) = e^x \quad e^{1/2} \quad x = \frac{1}{2}$

i) $P_4\left(\frac{1}{2}\right)$ using $M = 3 \quad (e < 3)$

$$|R_4| \le \left| \frac{3}{5!}\left(\frac{1}{2}\right)^5 \right| = \frac{3}{32 \cdot 5!} = \frac{3}{120 \cdot 32} = \frac{1}{40 \cdot 32} = \frac{1}{1280}$$

ii) ERROR $< \frac{1}{10000}$ again using $M = 3$ $(e < 3)$

$$\left| \frac{3}{(n+1)!}\left(\frac{1}{2}\right)^{n+1} \right| < \frac{1}{10000}$$

$$\frac{3}{2 \cdot 2^n \cdot (n+1)!} < \frac{1}{10000}$$

$$\frac{1}{2^n (n+1)!} < \frac{2}{30000} = \frac{1}{15,000}$$

$\boxed{\therefore n = 5 \text{ WORKS}}$

$2^n (n+1)!$

$n = 4 \quad 16 \cdot 5! = 1920$

$n = 5 \quad 32 \cdot 6! = 23040$

$$\frac{1}{23040} < \frac{1}{15000}$$

c) $f(x) = \cos x \qquad \cos 40° = \cos(45° - 5°)$

$\qquad = \cos\left(\frac{\pi}{4} - \underline{\underline{\frac{\pi}{36}}}\right) \quad 5° \frac{\pi}{180}$

distance from $\frac{\pi}{4}$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(\pi/4) = \frac{-\sqrt{2}}{2}$

$f''(\pi/4) = -\frac{\sqrt{2}}{2}$

$f'''\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$$|R_2| \le \left| \frac{1}{3!}\left(\frac{\pi}{36}\right)^3 \right|$$

$$= \frac{\pi^3}{36^3 \cdot 9}$$

$M = 1$

max sine value

ii) $\frac{1}{(n+1)!}\left(\frac{\pi}{36}\right)^{n+1} < \frac{5}{1000000} = \frac{1}{200,000}$ $\qquad \frac{\pi^{n+1}}{(n+1)!\, 36^{n+1}}$

$.000005$

$n = 3 \quad 2.416 \times 10^{-6}$

$\boxed{n = 3 \text{ WORKS}}$