ODE Cheat Sheet

First Order Equations

Separable

$y'(x) = f(x)g(y)$
$\int \frac{dy}{g(y)} = \int f(x)\,dx + C$

Linear First Order

$y'(x) + p(x)y(x) = f(x)$
$\mu(x) = \exp \int^x p(\xi)\,d\xi$ Integrating factor.
$(\mu y)' = f\mu$ Exact Derivative.
Solution: $y(x) = \frac{1}{\mu(x)}\left(\int f(\xi)\mu(\xi)\,d\xi + C\right)$

Exact

$0 = M(x,y)\,dx + N(x,y)\,dy$
Solution: $u(x,y) = \text{const}$ where
$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$
$\frac{\partial u}{\partial x} = M(x,y), \quad \frac{\partial u}{\partial y} = N(x,y)$
Condition: $M_y = N_x$

Non-Exact Form

$\mu(x,y)\,(M(x,y)\,dx + N(x,y)\,dy) = du(x,y)$
$M_y = N_x$
$N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y} = \mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right).$

Special cases

If $\frac{M_y - N_x}{M} = h(y)$, then $\mu(y) = \exp \int h(y)\,dy$
If $\frac{M_y - N_x}{N} = -h(x)$, then $\mu(y) = \exp \int h(x)\,dx$

Second Order Equations

Linear

$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x)$
$y(x) = y_h(x) + y_p(x)$
$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$

Constant Coefficients

$ay''(x) + by'(x) + cy(x) = f(x)$
$y(x) = e^{rx} \Rightarrow ar^2 + br + c = 0$

Cases

Distinct, real roots: $r = r_{1,2}$, $y_h(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
One real root: $y_h(x) = (c_1 + c_2 x)e^{rx}$
Complex roots: $r = \alpha \pm i\beta$, $y_h(x) = (c_1 \cos\beta x + c_2 \sin\beta x)e^{\alpha x}$

Cauchy-Euler Equations

$ax^2 y''(x) + bxy'(x) + cy(x) = f(x)$
$y(x) = x^r \Rightarrow ar(r-1) + br + c = 0$

Cases

Distinct, real roots: $r = r_{1,2}$, $y_h(x) = c_1 x^{r_1} + c_2 x^{r_2}$
One real root: $y_h(x) = (c_1 + c_2 \ln|x|)x^r$
Complex roots: $r = \alpha \pm i\beta$,
$y_h(x) = (c_1 \cos(\beta \ln|x|) + c_2 \sin(\beta \ln|x|))x^\alpha$

Nonhomogeneous Problems

Method of Undetermined Coefficients

$f(x)$	$y_p(x)$
$a_n x^n + \cdots + a_1 x + a_0$	$A_n x^n + \cdots + A_1 x + A_0$
ae^{bx}	Ae^{bx}
$a\cos\omega x + b\sin\omega x$	$A\cos\omega x + B\sin\omega x$

Modified Method of Undetermined Coefficients: if any term in the guess $y_p(x)$ is a solution of the homogeneous equation, then multiply the guess by x^k, where k is the smallest positive integer such that no term in $x^k y_p(x)$ is a solution of the homogeneous problem.

Reduction of Order

Homogeneous Case

Given $y_1(x)$ satisfies $L[y] = 0$, find second linearly independent solution as $v(x) = v(x)y_1(x)$. $z = v'$ satisfies a separable ODE.

Nonhomogeneous Case

Given $y_1(x)$ satisfies $L[y] = 0$, find solution of $L[y] = f$ as $v(x) = v(x)y_1(x)$. $z = v'$ satisfies a first order linear ODE.

Method of Variation of Parameters

$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$
$c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0$
$c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = \frac{f(x)}{a(x)}$

Applications

Free Fall

$x''(t) = -g$
$v'(t) = -g + f(v)$

Population Dynamics

$P'(t) = kP(t)$
$P'(t) = kP(t) - bP^2(t)$

Newton's Law of Cooling

$T'(t) = -k(T(t) - T_a)$

Oscillations

$mx''(t) + kx(t) = 0$
$mx''(t) + bx'(t) + kx(t) = 0$
$mx''(t) + bx'(t) + kx(t) = F(t)$

Types of Damped Oscillation

Overdamped, $b^2 > 4mk$
Critically Damped, $b^2 = 4mk$
Underdamped, $b^2 < 4mk$

Numerical Methods

Euler's Method

$y_0 = y(x_0)$,
$y_n = y_{n-1} + \Delta x f(x_{n-1}, y_{n-1}), \quad n = 1, \ldots, N.$

Series Solutions

Taylor Method

$f(x) \sim \sum_{n=0}^{\infty} c_n x^n$, $c_n = \frac{f^{(n)}(0)}{n!}$

1. Differentiate DE repeatedly.
2. Apply initial conditions.
3. Find Taylor coefficients.
4. Insert coefficients into series form for $y(x)$.

Power Series Solution

1. Let $y(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$.
2. Find $y'(x)$, $y''(x)$.
3. Insert expansions in DE.
4. Collect like terms using reindexing.
5. Find recurrence relation.
6. Solve for coefficients and insert in $y(x)$ series.

Ordinary and Singular Points

$y'' + a(x)y' + b(x)y = 0$. x_0 is a
Ordinary point: $a(x), b(x)$ real analytic in $|x - x_0| < R$
Regular singular point: $(x - x_0)a(x)$, $(x - x_0)^2 b(x)$ have convergent Taylor series about $x = x_0$.
Irregular singular point: Not ordinary or regular singular point.

Frobenius Method

1. Let $y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$.
2. Obtain indicial equation $r(r-1) + a_0 r + b_0$.
3. Find recurrence relation based on types of roots of indicial equation.
4. Solve for coefficients and insert in $y(x)$ series.

Laplace Transforms

Transform Pairs

c	$\frac{c}{s}$
e^{at}	$\frac{1}{s-a}, \quad s > a$
t^n	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin\omega t$	$\frac{\omega}{s^2 + \omega^2}$
$\cos\omega t$	$\frac{s}{s^2 + \omega^2}$
$\sinh at$	$\frac{a}{s^2 - a^2}$
$\cosh at$	$\frac{s}{s^2 - a^2}$
$H(t-a)$	$\frac{e^{-as}}{s}, \quad s > 0$
$\delta(t-a)$	$e^{-as}, \quad a \geq 0, s > 0$

Laplace Transform Properties

$\mathcal{L}[af(t)+bg(t)] = aF(s)+bG(s)$

$\mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$

$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s)-f(0)$

$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s)-sf(0)-f'(0)$

$\mathcal{L}[e^{at}f(t)] = F(s-a)$

$\mathcal{L}[H(t-a)f(t-a)] = e^{-as}F(s)$

$\mathcal{L}[(f*g)(t)] = \mathcal{L}[\int_0^t f(t-u)g(u)\,du] = F(s)G(s)$

Solve Initial Value Problem

1. Transform DE using initial conditions.
2. Solve for $Y(s)$.
3. Use transform pairs, partial fraction decomposition, to obtain $y(t)$.

Special Functions

Legendre Polynomials

$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$

$(1-x^2)y''-2xy'+n(n+1)y=0.$

$(n+1)P_{n+1}(x) = (2n+1)xP_n(x)-nP_{n-1}(x), \quad n=1,2,\ldots.$

$g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty}P_n(x)t^n, \quad |x|\le 1, |t|<1.$

Bessel Functions, $J_p(x)$, $N_p(x)$

$x^2y''+xy'+(x^2-p^2)y=0.$

Gamma Functions

$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt, \quad x>0.$

$\Gamma(x+1) = x\Gamma(x).$

Systems of Differential Equations

Planar Systems

$x' = ax+by$

$y' = cx+dy.$

$x''-(a+d)x'+(ad-bc)x=0.$

Matrix Form

$\mathbf{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \equiv A\mathbf{x}.$

Guess $\mathbf{x} = \mathbf{v}e^{\lambda t} \Rightarrow A\mathbf{v} = \lambda\mathbf{v}.$

Eigenvalue Problem

$A\mathbf{v} = \lambda\mathbf{v}.$

Find Eigenvalues: $\det(A-\lambda I)=0$

Find Eigenvectors $(A-\lambda I)\mathbf{v}=0$ for each λ.

Cases

Real, Distinct Eigenvalues: $\mathbf{x}(t) = c_1e^{\lambda_1 t}\mathbf{v}_1 + c_2e^{\lambda_2 t}\mathbf{v}_2$

Repeated Eigenvalue: $\mathbf{x}(t) = c_1e^{\lambda t}\mathbf{v}_1 + c_2e^{\lambda t}(\mathbf{v}_2+t\mathbf{v}_1)$, where $A\mathbf{v}_2-\lambda\mathbf{v}_2 = \mathbf{v}_1$ for $\mathbf{v}_2$.

Complex Conjugate Eigenvalues: $\mathbf{x}(t) = c_1 Re(e^{\alpha t}(\cos\beta t + i\sin\beta t)\mathbf{v}) + c_2 Im(e^{\alpha t}(\cos\beta t + i\sin\beta t)\mathbf{v}).$

Solution Behavior

Stable Node: $\lambda_1,\lambda_2<0$.

Unstable Node: $\lambda_1,\lambda_2>0$.

Saddle: $\lambda_1\lambda_2<0$.

Center: $\lambda = i\beta$.

Stable Focus: $\lambda = \alpha+i\beta$, $\alpha<0$.

Unstable Focus: $\lambda = \alpha+i\beta$, $\alpha>0$.

Matrix Solutions

Let $\mathbf{x}' = A\mathbf{x}$.

Find eigenvalues λ_i

Find eigenvectors $\mathbf{v}_i = \begin{pmatrix} v_{i1} \\ v_{i2} \end{pmatrix}$

Form the Fundamental Matrix Solution:

$\Phi = \begin{pmatrix} v_{11}e^{\lambda_1 t} & v_{21}e^{\lambda_2 t} \\ v_{12}e^{\lambda_1 t} & v_{22}e^{\lambda_2 t} \end{pmatrix}$

General Solution: $\mathbf{x}(t) = \Phi(t)\mathbf{C}$ for $\mathbf{C}$

Find C: $\mathbf{x}_0 = \Phi(t_0)\mathbf{C} \Rightarrow \mathbf{C} = \Phi^{-1}(t_0)\mathbf{x}_0$

Particular Solution: $\mathbf{x}(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0$.

Principal Matrix solution: $\Psi(t) = \Phi(t)\Phi^{-1}(t_0)$.

Particular Solution: $\mathbf{x}(t) = \Psi(t)\mathbf{x}_0$.

Note: $\Psi' = A\Psi, \quad \Psi(t_0) = I$.

Nonhomogeneous Matrix Solutions

$\mathbf{x}(t) = \Phi(t)\mathbf{C} + \Phi(t)\int_{t_0}^t \Phi^{-1}(s)\mathbf{f}(s)\,ds$

$\mathbf{x}(t) = \Psi(t)\mathbf{x}_0 + \Psi(t)\int_{t_0}^t \Psi^{-1}(s)\mathbf{f}(s)\,ds$

2×2 Matrix Inverse

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$