Proof: The proof will use the original principle of induction (Theorem 10.2). For each $r \in \mathbb{N}$, let $Q(r)$ be the statement "$P(r + m - 1)$ is true." Then from (a) we know that $Q(1)$ holds. Now let $j \in \mathbb{N}$ and suppose that $Q(j)$ holds. That is, $P(j + m - 1)$ is true. Since $j \in \mathbb{N}$,

$$j + m - 1 = m + (j - 1) \geq m,$$

so by (b), $P(j + m)$ must be true. Thus $Q(j + 1)$ holds and the induction step is verified. We conclude that $Q(r)$ holds for all $r \in \mathbb{N}$.

Now if $n \geq m$, let $r = n - m + 1$, so that $r \in \mathbb{N}$. Since $Q(r)$ holds, $P(r + m - 1)$ is true. But $P(r + m - 1)$ is the same as $P(n)$, so $P(n)$ is true for all $n \geq m$. ♦

ANSWERS TO PRACTICE PROBLEMS

10.5 The general formula is

$$1+3+5+\cdots+(2n-1) = n^2,$$

and we have already seen that this is true for $n = 1$. For the induction step, suppose that $1 + 3 + 5 + \cdots + (2k - 1) = k^2$. Then

$$\begin{aligned} 1+3+5+\cdots+(2k-1)+(2k+1) &= k^2+(2k+1) \\ &= (k+1)^2. \end{aligned}$$

Since this is the formula for $n = k + 1$, we conclude by induction that the formula holds for all $n \in \mathbb{N}$. ♦

EXERCISES

10.1 Mark each statement True or False. Justify each answer.

(a) If S is a nonempty subset of $\mathbb{N}$, then there exists an element $m \in S$ such that $m \geq k$ for all $k \in S$.

(b) The Principle of Mathematical Induction enables us to prove that a statement is true for all natural numbers without directly verifying it for each number.

10.2 Mark each statement True or False. Justify each answer.

(a) A proof using mathematical induction consists of two parts: establishing the basis for induction and verifying the induction hypothesis.

(b) Suppose m is a natural number greater than 1. To prove $P(k)$ is true for all $k \geq m$, we must first show that $P(k)$ is false for all k such that $1 \leq k < m$.

*10.3 Prove that $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all $n \in \mathbb{N}$.

*10.4 Prove that $1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$ for all $n \in \mathbb{N}$.

10.5 Prove that $1^3 + 2^3 + \cdots + n^3 = (1+2+\cdots+n)^2$ for all $n \in \mathbb{N}$.

*10.6 Prove that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}, \text{ for all } n \in \mathbb{N}.$$

*10.7 Prove that $1 + r + r^2 + \cdots + r^n = (1 - r^{n+1})/(1 - r)$ for all $n \in \mathbb{N}$, when $r \neq 1$.

*10.8 Prove that

$$\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \cdots + \frac{1}{4n^2 - 1} = \frac{n}{2n+1}, \text{ for all } n \in \mathbb{N}.$$

10.9 Prove that $5^{2n} - 1$ is a multiple of 8 for all $n \in \mathbb{N}$.

10.10 Prove that $9^n - 4^n$ is a multiple of 5 for all $n \in \mathbb{N}$.

10.11 Conjecture a formula for the sum $5 + 9 + 13 + \cdots + (4n + 1)$, and prove your conjecture using mathematical induction.

10.12 Prove that $2 + 5 + 8 + \cdots + (3n - 1) = \frac{1}{2}n(3n+1)$ for all $n \in \mathbb{N}$.

10.13 Indicate for which natural numbers n the given inequality is true. Prove your answers by induction.

(a) $n^2 \leq n!$
(b) $n^2 \leq 2^n$
(c) $2^n \leq n!$

*10.14 Use induction to prove Bernoulli's inequality: If $1 + x > 0$, then $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

*10.15 Prove the principle of strong induction: Let $P(n)$ be a statement that is either true or false for each $n \in \mathbb{N}$. Then $P(n)$ is true for all $n \in \mathbb{N}$ provided that

(a) $P(1)$ is true, and
(b) for each $k \in \mathbb{N}$, if $P(j)$ is true for all integers j such that $1 \leq j \leq k$, then $P(k + 1)$ is true.

10.16 Indicate what is wrong with each of the following induction "proofs."

(a) **Theorem:** For each $n \in \mathbb{N}$, let $P(n)$ be the statement "Any collection of n marbles consists of marbles of the same color." Then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof: Clearly, $P(1)$ is a true statement. Now suppose that $P(k)$ is a true statement for some $k \in \mathbb{N}$. Let S be a collection of $k + 1$ marbles. If one marble, call it x, is removed, then the induction hypothesis applied to the remaining k marbles implies that these k marbles all have the same color. Call this color C. Now if x is returned to the set S and a different marble is removed, then again the remaining k marbles must all be of the same color C. But one of these marbles is x, so in fact all $k+1$ marbles have the same color C. Thus $P(k+1)$ is true, and by induction we conclude that $P(n)$ is true for all $n \in \mathbb{N}$. ♦

(b) **Theorem:** For each $n \in \mathbb{N}$, let $P(n)$ be the statement "$n^2 + 7n + 3$ is an even integer." Then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof: Suppose that $P(k)$ is true for some $k \in \mathbb{N}$. That is, $k^2 + 7k + 3$ is an even integer. But then

$$\begin{aligned}(k+1)^2 + 7(k+1) + 3 &= (k^2 + 2k + 1) + 7k + 7 + 3 \\ &= (k^2 + 7k + 3) + 2(k+4),\end{aligned}$$

and this number is even, since it is the sum of two even numbers. Thus $P(k+1)$ is true. We conclude by induction that $P(n)$ is true for all $n \in \mathbb{N}$. ♦

10.17 In the song "The Twelve Days of Christmas," gifts are sent on successive days according to the following pattern:

First day: A partridge in a pear tree.
Second day: Two turtledoves and another partridge.
Third day: Three French hens, two turtledoves, and a partridge.
And so on.

For each $i = 1, \ldots, 12$, let g_i be the number of gifts sent on the ith day. Then $g_1 = 1$, and for $i = 2, \ldots, 12$ we have

$$g_i = g_{i-1} + i.$$

Now let t_n be the total number of gifts sent during the first n days of Christmas. Find a formula for t_n in the form

$$t_n = \frac{n(n+a)(n+b)}{c},$$

where $a, b, c \in \mathbb{N}$.

10.18 Let $0! = 1$ and for $n \in \mathbb{N}$ define $n!$ (read "n factorial") by

$$n! = n[(n-1)!].$$

Then let

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \quad \text{for } r = 0, 1, 2, \ldots, n.$$

(a) Show that

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r} \quad \text{for } r = 0, 1, 2, \ldots, n.$$

*(b) Use part (a) and mathematical induction to prove the **binomial theorem**:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n}b^n$$
$$= a^n + na^{n-1}b + \frac{1}{2}n(n-1)a^{n-2}b^2 + \cdots + nab^{n-1} + b^n.$$

10.19 Prove Theorem 10.6 by using the well-ordering property of $\mathbb{N}$ instead of the principle of mathematical induction.

10.20 Use the principle of mathematical induction to prove the well-ordering property of $\mathbb{N}$. Thus we could have taken Theorem 10.2 as an axiom and derived 10.1 as a theorem.

Exercise 10.21 illustrates how the basic properties of addition of natural numbers can be derived from a few simple axioms. These axioms are called the Peano axioms in honor of the Italian mathematician Giuseppe Peano, who developed this approach in the late nineteenth century. We suppose that there exist a set P whose elements are called **natural numbers** and a relation of **successor** with the following properties:

P1. There exists a natural number, denoted by 1, that is not the successor of any other natural number.

P2. Every natural number has a unique successor. If $m \in P$, then we let m' denote the successor of m.

P3. Every natural number except 1 is the successor of exactly one natural number.

P4. If M is a set of natural numbers such that
 (i) $1 \in M$ and
 (ii) for each $k \in P$, if $k \in M$, then $k' \in M$,
then $M = P$.

Axioms P1 to P3 express the intuitive notion that 1 is the first natural number and that we can progress through the natural numbers in succession one at a time. Axiom P4 is the equivalent of the principle of mathematical induction. Using these axioms, we can define what addition means. We begin by defining what it means to add 1.

D1. For every $n \in P$, define $n + 1 = n'$.

That is, $n+1$ is the unique successor of n whose existence is guaranteed by axiom P2. Following this pattern, it is clear that we want to define $n+2 = (n+1)+1$, $n+3 = [(n+1)+1]+1$, and so on. To define $n+m$ for all $m \in P$, we use a recursive definition:

D2. Let $n, m \in P$. If $m = k'$ and $n + k$ is defined, then define $n+m$ to be $(n+k)'$.

EXERCISES

11.1 Mark each statement True or False. Justify each answer.

(a) Axioms A1 to A5, M1 to M5, and DL describe an algebraic system known as a field.

(b) The property that $x + y = y + x$ for all $x, y \in \mathbb{R}$ is called an associative law.

(c) If $x, y, z \in \mathbb{R}$ and $x < y$, then $xz < yz$.

11.2 Mark each statement True or False. Justify each answer.

(a) Axioms A1 to A5, M1 to M5, DL, and O1 to O4 describe an algebraic system known as an ordered field.

(b) If $x, y \in \mathbb{R}$ and $x < y + \varepsilon$ for every $\varepsilon > 0$, then $x < y$.

(c) If $x, y \in \mathbb{R}$, then $|x + y| \geq |x| + |y|$.

11.3 Let x, y, and z be real numbers. Prove the following.

(a) $-(-x) = x$.

(b) $(-x) \cdot y = -(xy)$ and $(-x) \cdot (-y) = xy$.

(c) If $x \neq 0$, then $(1/x) \neq 0$ and $1/(1/x) = x$.

(d) If $x \cdot z = y \cdot z$ and $z \neq 0$, then $x = y$.

(e) If $x \neq 0$, then $x^2 > 0$.

(f) $0 < 1$.

(g) If $x > 0$, then $1/x > 0$. If $x < 0$, then $1/x < 0$.

(h) If $0 < x < y$, then $0 < 1/y < 1/x$.

(i) If $xy > 0$, then either (i) $x > 0$ and $y > 0$, or (ii) $x < 0$ and $y < 0$.

(j) For each $n \in \mathbb{N}$, if $0 < x < y$, then $x^n < y^n$.

(k) If $0 < x < y$, then $0 < \sqrt{x} < \sqrt{y}$.

11.4 Prove: If $x \geq 0$ and $x \leq \varepsilon$ for all $\varepsilon > 0$, then $x = 0$.

11.5 Prove Theorem 11.9(c): $|xy| = |x| \cdot |y|$.

***11.6** (a) Prove: $\big||x| - |y|\big| \leq |x - y|$.

(b) Prove: If $|x - y| < c$, then $|x| < |y| + c$.

***11.7** Suppose that $x_1, x_2, \ldots, x_n$ are real numbers. Prove that

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

11.8 Let $P = \{x \in \mathbb{R}: x > 0\}$. Show that P satisfies the following:

(a) If $x, y \in P$, then $x + y \in P$.

(b) If $x, y \in P$, then $x \cdot y \in P$.

(c) For each $x \in \mathbb{R}$, exactly one of the following three statements is true: $x \in P$, $x = 0$, $-x \in P$.

11.9 Let F be a field and suppose that P is a subset of F that satisfies the three properties in Exercise 11.8. Define $x < y$ iff $y - x \in P$. Prove that "$<$" satisfies axioms O1, O2, and O3. Thus in defining an ordered field, either we can begin with the properties of "<" as in the text, or we can begin by identifying a certain subset as "positive."

11.10 Prove that in any ordered field F, $a^2 + 1 > 0$ for all $a \in F$. Conclude from this that if the equation $x^2 + 1 = 0$ has a solution in a field, then that field cannot be ordered. (Thus it is not possible to define an order relation on the set of all complex numbers that will make it an ordered field.)

11.11 Let $\mathbb{F}$ be the field of rational functions described in Example 11.5.

(a) Show that the ordering given there satisfies the order axioms O1, O2, and O3.

(b) Write the following polynomials in order of increasing size:

$$x^2,\ -x^3,\ 5,\ x+2,\ 3-x.$$

(c) Write the following functions in order of increasing size:

$$\frac{x^2+2}{x-1},\ \frac{x^2-2}{x+1},\ \frac{x+1}{x^2-2},\ \frac{x+2}{x^2-1}.$$

11.12 To actually construct the rationals $\mathbb{Q}$ from the integers $\mathbb{Z}$, let $S = \{(a,b)\colon a, b \in \mathbb{Z} \text{ and } b \neq 0\}$. Define an equivalence relation "~" on S by $(a,b) \sim (c,d)$ iff $ad = bc$. We then define the set $\mathbb{Q}$ of rational numbers to be the set of equivalence classes corresponding to ~. The equivalence class determined by the ordered pair (a,b) we denote by $[a/b]$. Then $[a/b]$ is what we usually think of as the fraction a/b. For $a, b, c, d \in \mathbb{Z}$ with $b \neq 0$ and $d \neq 0$, we define addition and multiplication in $\mathbb{Q}$ by

$$[a/b]+[c/d] = [(ad+bc)/bd],$$
$$[a/b]\cdot[c/d] = [(ac)/(bd)].$$

We say that $[a/b]$ is *positive* if $ab \in \mathbb{N}$. Since $a, b \in \mathbb{Z}$ with $b \neq 0$, this is equivalent to requiring $ab > 0$. The set of positive rationals is denoted by $\mathbb{Q}^+$, and we define an order "<" on $\mathbb{Q}$ by

$$x < y \quad \text{iff} \quad y - x \in \mathbb{Q}^+.$$

(a) Verify that ~ is an equivalence relation on S.

(b) Show that addition and multiplication are well-defined. That is, suppose $[a/b] = [p/q]$ and $[c/d] = [r/s]$. Show that $[(ad+bc)/bd] = [(ps+qr)/qs]$ and $[ac/bd] = [pr/qs]$.

(c) For any $b \in \mathbb{Z} \setminus \{0\}$, show that $[0/b] = [0/1]$ and $[b/b] = [1/1]$.

(d) For any $a, b \in \mathbb{Z}$ with $b \neq 0$, show that $[a/b] + [0/1] = [a/b]$ and $[a/b]\cdot[1/1] = [a/b]$. Thus $[0/1]$ corresponds to zero and $[1/1]$ corresponds to 1.

(e) For any $a, b \in \mathbb{Z}$ with $b \neq 0$, show that $[a/b] + [(-a)/b] = [0/1]$ and $[a/b]\cdot[b/a] = [1/1]$.

ANSWERS TO PRACTICE PROBLEMS

12.4 Any real number x such that $x^2 \geq 2$ is an upper bound for T. The smallest of these upper bounds is $\sqrt{2}$, but since $\sqrt{2} \notin \mathbb{Q}$, set T has no maximum. The minimum of T is 0. Any real x such that $x \leq 0$ is a lower bound.

12.6 $m = \inf S$ iff (i) $m \leq s$, for all $s \in S$, and (ii) if $m' > m$, then there exists $s' \in S$ such that $s' < m'$.

12.13 Since x is rational and $x \neq 0$, we have $x = m/n$ for some nonzero integers m and n. If xy were rational, then we could write $xy = p/q$ for some $p, q \in \mathbb{Z}$. But then

$$y = \frac{xy}{x} = \frac{p/q}{m/n} = \frac{pn}{mq},$$

so y would have to be rational too, a contradiction.

EXERCISES

12.1 Mark each statement True or False. Justify each answer.

(a) If a nonempty subset of $\mathbb{R}$ has an upper bound, then it has a least upper bound.
(b) If a nonempty subset of $\mathbb{R}$ has an infimum, then it is bounded.
(c) Every nonempty bounded subset of $\mathbb{R}$ has a maximum and a minimum.
(d) If m is an upper bound for S and $m' < m$, then m' is not an upper bound for S.
(e) If $m = \inf S$ and $m' < m$, then m' is a lower bound for S.

12.2 Mark each statement True or False. Justify each answer.

(a) For each real number x and each $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $n\varepsilon > x$.
(b) If x and y are irrational, then xy is irrational.
(c) Between any two unequal rational numbers there is an irrational number.
(d) Between any two unequal irrational numbers there is a rational number.
(e) The rational and irrational numbers alternate, one then the other.

12.3 For each subset of $\mathbb{R}$, give its supremum and its maximum, if they exist. Otherwise, write "none."

(a) $\{1, 3\}$ (b) $\{\pi, 3\}$

(c) $[0, 4]$ (d) $(0, 4)$

(e) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ (f) $\left\{1-\frac{1}{n} : n \in \mathbb{N}\right\}$

(g) $\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$ (h) $\left\{(-1)^n\left(1+\frac{1}{n}\right) : n \in \mathbb{N}\right\}$

(i) $\left\{n+\frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$ (j) $(-\infty, 4)$

(k) $\bigcap_{n=1}^{\infty}\left(1-\frac{1}{n}, 1+\frac{1}{n}\right)$ (l) $\bigcup_{n=1}^{\infty}\left[\frac{1}{n}, 2-\frac{1}{n}\right]$

(m) $\{r \in \mathbb{Q} : r < 5\}$ (n) $\{r \in \mathbb{Q} : r^2 \le 5\}$

12.4 Repeat Exercise 12.3 for the infimum and the minimum of each set.

12.5 Let S be a nonempty bounded subset of $\mathbb{R}$ and let $m = \sup S$. Prove that $m \in S$ iff $m = \max S$.

✓**12.6** Let S be a nonempty bounded subset of $\mathbb{R}$. Prove that $\sup S$ is unique.

***12.7** Let S be a nonempty bounded subset of $\mathbb{R}$ and let $k \in \mathbb{R}$. Define $kS = \{ks : s \in S\}$. Prove the following:

(a) If $k \ge 0$, then $\sup(kS) = k \cdot \sup S$ and $\inf(kS) = k \cdot \inf S$.
(b) If $k < 0$, then $\sup(kS) = k \cdot \inf S$ and $\inf(kS) = k \cdot \sup S$.

12.8 Let S and T be nonempty bounded subsets of $\mathbb{R}$ with $S \subseteq T$. Prove that $\inf T \le \inf S \le \sup S \le \sup T$.

12.9 (a) Prove: If $y > 0$, then there exists $n \in \mathbb{N}$ such that $n-1 \le y < n$.
(b) Prove that the n in part (a) is unique.

12.10 (a) Prove: If x and y are real numbers with $x < y$, then there are infinitely many rational numbers in the interval $[x, y]$.
(b) Repeat part (a) for irrational numbers.

12.11 Let y be a positive real number. Prove that for every $n \in \mathbb{N}$ there exists a unique positive real number x such that $x^n = y$.

***12.12** Let D be a nonempty set and suppose that $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$. Define the function $f+g: D \to \mathbb{R}$ by $(f+g)(x) = f(x) + g(x)$.

(a) If $f(D)$ and $g(D)$ are bounded above, then prove that $(f+g)(D)$ is bounded above and $\sup[(f+g)(D)] \le \sup f(D) + \sup g(D)$.
(b) Find an example to show that a strict inequality in part (a) may occur.
(c) State and prove the analog of part (a) for infima.

12.13 Let $x \in \mathbb{R}$. Prove that $x = \sup\{q \in \mathbb{Q} : q < x\}$.

12.14 Let a/b be a fraction in lowest terms with $0 < a/b < 1$.

(a) Prove that there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n+1} \leq \frac{a}{b} < \frac{1}{n}.$$

(b) If n is chosen as in part (a), prove that $a/b - 1/(n+1)$ is a fraction that in lowest terms has a numerator less than a.

(c) Use part (b) and the principle of strong induction (Exercise 10.15) to prove that a/b can be written as a finite sum of distinct unit fractions:

$$\frac{a}{b} = \frac{1}{n_1} + \cdots + \frac{1}{n_k},$$

where $n_1, \ldots, n_k \in \mathbb{N}$. (As a point of historical interest, we note that in the ancient Egyptian system of arithmetic all fractions were expressed as sums of unit fractions and then manipulated using tables.)

12.15 Prove Euclid's division algorithm: If a and b are natural numbers, then there exist unique numbers q and r, each of which is either 0 or a natural number, such that $r < a$ and $b = qa + r$.

12.16 Let $\mathbb{F}$ be the ordered field of rational functions as given in Example 11.5, and note that $\mathbb{F}$ contains both $\mathbb{N}$ and $\mathbb{R}$ as subsets.

(a) Show that $\mathbb{F}$ does not have the Archimedean property. That is, find a member z in $\mathbb{F}$ such that $z > n$ for every $n \in \mathbb{N}$.

(b) Show that the property in Theorem 12.10(c) does not apply. That is, find a positive member z in $\mathbb{F}$ such that, for all $n \in \mathbb{N}$, $0 < z \leq 1/n$.

(c) Show that $\mathbb{F}$ does not satisfy the completeness axiom. That is, find a subset B of $\mathbb{F}$ such that B is bounded above, but B has no least upper bound. Justify your answer.

12.17 We have said that the real numbers can be characterized as a complete ordered field. This means that any other complete ordered field F is essentially the same as $\mathbb{R}$ in the sense that there exists a bijection $f: \mathbb{R} \to F$ with the following properties for all $a, b \in \mathbb{R}$.

(1) $f(a + b) = f(a) + f(b)$
(2) $f(a \cdot b) = f(a) \cdot f(b)$
(3) $a < b$ iff $f(a) < f(b)$

(Such a function is called an **order isomorphism**.) We can construct the function f by first defining $f(0) = 0_F$ and $f(1) = 1_F$, where 0_F and 1_F are the unique elements of F given in axioms A4 and M4. Then define $f(n+1) = f(n) + 1_F$ and $f(-n) = -f(n)$ for all $n \in \mathbb{N}$. This extends the domain of f to all of $\mathbb{Z}$.

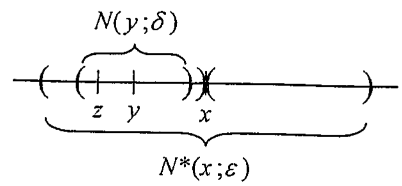

Figure 13.2

ANSWERS TO PRACTICE PROBLEMS

13.5 int $S = (1,2) \cup (2,3)$ and bd $S = \{1,2,3\}$.

13.9 The empty set $\varnothing$ is both open and closed, since it is the complement of the set $\mathbb{R}$, which is both open and closed. Or, to put it another way, $\varnothing$ is open since int $\varnothing = \varnothing$, and $\varnothing$ is closed since bd $\varnothing = \varnothing \subseteq \varnothing$.

13.13 There are many possibilities. For a simple one, let $A_n = [1/n, 2]$ for all $n \in \mathbb{N}$. Then $\bigcup_{n=1}^{\infty} A_n = (0, 2]$, which is not closed.

EXERCISES

13.1 Let $S \subseteq \mathbb{R}$. Mark each statement True or False. Justify each answer.

(a) int $S \cap$ bd $S = \varnothing$
(b) int $S \subseteq S$
(c) bd $S \subseteq S$
(d) S is open iff $S =$ int S.
(e) S is closed iff $S =$ bd S.
(f) If $x \in S$, then $x \in$ int S or $x \in$ bd S.
(g) Every neighborhood is an open set.
(h) The union of any collection of open sets is open.
(i) The union of any collection of closed sets is closed.

13.2 Let $S \subseteq \mathbb{R}$. Mark each statement True or False. Justify each answer.

(a) bd $S =$ bd $(\mathbb{R} \backslash S)$
(b) bd $S \subseteq \mathbb{R} \backslash S$
(c) $S \subseteq S' \subseteq$ cl S
(d) S is closed iff cl $S \subseteq S$.
(e) S is closed iff $S' \subseteq S$.
(f) If $x \in S$ and x is not an isolated point of S, then $x \in S'$.

(g) The set $\mathbb{R}$ of real numbers is neither open nor closed.
(h) The intersection of any collection of open sets is open.
(i) The intersection of any collection of closed sets is closed.

13.3 Find the interior of each set.

(a) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$
(b) $[0,3] \cup (3,5)$
(c) $\left\{r \in \mathbb{Q} : 0 < r < \sqrt{2}\right\}$
(d) $\left\{r \in \mathbb{Q} : r \geq \sqrt{2}\right\}$
(e) $[0,2] \cap [2,4]$

13.4 Find the boundary of each set in Exercise 13.3.

13.5 Classify each of the following sets as open, closed, neither, or both.

(a) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$
(b) $\mathbb{N}$
(c) $\mathbb{Q}$
(d) $\bigcap_{n=1}^{\infty}\left(0, \frac{1}{n}\right)$
(e) $\left\{x : |x-5| \leq \frac{1}{2}\right\}$
(f) $\{x : x^2 > 0\}$

13.6 Find the closure of each set in Exercise 13.5.

13.7 If A is open and B is closed, prove that $A \backslash B$ is open and $B \backslash A$ is closed.

13.8 Prove: For each $x \in \mathbb{R}$ and $\varepsilon > 0$, $N^*(x; \varepsilon)$ is an open set.

13.9 Prove: $(\operatorname{cl} S) \backslash (\operatorname{int} S) = \operatorname{bd} S$.

13.10 Let S be a bounded infinite set and let $x = \sup S$. Prove: If $x \notin S$, then $x \in S'$.

***13.11** Prove: If x is an accumulation point of the set S, then every neighborhood of x contains infinitely many points of S.

13.12 (a) Prove: $\operatorname{bd} S = (\operatorname{cl} S) \cap [\operatorname{cl}(\mathbb{R} \backslash S)]$.
(b) Prove: $\operatorname{bd} S$ is a closed set.

13.13 Prove: S' is a closed set.

13.14 Prove Theorem 13.17(c) and (d).

13.15 Let A be a nonempty open subset of $\mathbb{R}$ and let $\mathbb{Q}$ be the set of rationals. Prove that $A \cap \mathbb{Q} \neq \varnothing$.

13.16 Let S and T be subsets of $\mathbb{R}$. Prove the following.

(a) $\text{cl}\,(\text{cl}\,S) = \text{cl}\,S$
(b) $\text{cl}\,(S \cup T) = (\text{cl}\,S) \cup (\text{cl}\,T)$
(c) $\text{cl}\,(S \cap T) \subseteq (\text{cl}\,S) \cap (\text{cl}\,T)$
(d) Find an example to show that equality need not hold in part (c).

13.17 Let S and T be subsets of $\mathbb{R}$. Prove the following.

(a) $\text{int}\,S$ is an open set.
(b) $\text{int}\,(\text{int}\,S) = \text{int}\,S$
(c) $\text{int}\,(S \cap T) = (\text{int}\,S) \cap (\text{int}\,T)$
(d) $(\text{int}\,S) \cup (\text{int}\,T) \subseteq \text{int}\,(S \cup T)$
(e) Find an example to show that equality need not hold in part (d).

13.18 For any set $S \subseteq \mathbb{R}$, let $\overline{S}$ denote the intersection of all the closed sets containing S.

(a) Prove that $\overline{S}$ is a closed set.
(b) Prove that $\overline{S}$ is the smallest closed set containing S. That is, show that $S \subseteq \overline{S}$, and if C is any closed set containing S, then $\overline{S} \subseteq C$.
(c) Prove that $\overline{S} = \text{cl}\,S$.
(d) If S is bounded, prove that $\overline{S}$ is bounded.

13.19 For any set $S \subseteq \mathbb{R}$, let S° denote the union of all the open sets contained in S.

(a) Prove that S° is an open set.
(b) Prove that S° is the largest open set contained in S. That is, show that $S^\circ \subseteq S$, and if U is any open set contained in S, then $U \subseteq S^\circ$.
(c) Prove that $S^\circ = \text{int}\,S$.

13.20 In this exercise we outline a proof of the following theorem: "A subset of $\mathbb{R}$ is open iff it is the union of countably many disjoint open intervals in $\mathbb{R}$."

(a) Let S be a nonempty open subset of $\mathbb{R}$. For each $x \in S$, let $A_x = \{a \in \mathbb{R} : (a, x] \subseteq S\}$ and let $B_x = \{b \in \mathbb{R} : [x, b) \subseteq S\}$. Use the fact that S is open to show that A_x and B_x are both nonempty.
(b) If A_x is bounded below, let $a_x = \inf A_x$. Otherwise, let $a_x = -\infty$. If B_x is bounded above, let $b_x = \sup B_x$; otherwise, let $b_x = \infty$. Show that $a_x \notin S$ and $b_x \notin S$.
(c) Let I_x be the open interval (a_x, b_x). Clearly, $x \in I_x$. Show that $I_x \subseteq S$. (*Hint*: Consider two cases for $y \in I_x : y < x$ and $y > x$.)
(d) Show that $S = \bigcup_{x \in S} I_x$.
(e) Show that the intervals $\{I_x : x \in S\}$ are pairwise disjoint. That is, suppose $x, y \in S$ with $x \neq y$. If $I_x \cap I_y \neq \varnothing$, show that $I_x = I_y$.
(f) Show that the set of distinct intervals $\{I_x : x \in S\}$ is countable.