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Are operator algebras Banach algebras?

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ABSTRACT. We discuss some aspects of the current state of the study of algebras of operators on a Hilbert space, from the Banach algebraic perspective. The advent of operator space theory has given a recent impetus to the general theory of such operator algebras. To illustrate the situation, we focus on some recent results mainly obtained by the author and Le Merdy, and present several 'test questions', which point out the interactions between the Banach algebraic and the operator space structures involved.

1. Introduction

Operator algebras form a particularly nice class of Banach algebras, and constitute a pleasant setting for many Banach algebraic ideas. The advent of operator space theory has given a recent impetus to the *general theory* of nonselfadjoint operator algebras (see e.g. our forthcoming text [**4**] with Christian Le Merdy on operator algebras and their modules from an operator space perspective). With the operator space viewpoint in mind, we believe that now is a good time to look for fresh applications of Banach algebraic ideas to operator algebras. In the present article, we illustrate in a nontechnical way, and with some test questions, the current position that operator algebras hold amongst the Banach algebras[1]. In Section 2 we collect a group of related problems which it is about time were solved, many of which illustrate the question in our title. Some of these questions may even not be hard, with the right approach.

We begin with some background and notation. Throughout H is a (complex) Hilbert space. A *concrete operator space* is a (linear) subspace of $B(H)$. The set $M_n(B(H))$ of $n \times n$ operator matrices has a natural norm via the isomorphism $M_n(B(H)) \cong B(H^{(n)})$. An *abstract operator space* is a vector space X, with a

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[1]This article was first intended to be a summary of the talk we gave at this conference: a survey of our work on operator space multipliers, and their applications to operator algebras, particularly those of a Banach algebraic flavor. However in the interim we have summarized this work elsewhere. In particular, in the short survey [**6**], aimed at a general audience, the reader will find the essence of our Banach Algebras '03 talk.

norm $\|\cdot\|_n$ on $M_n(X)$ for all $n \in \mathbb{N}$, such that

$$\left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\|_{n+m} = \max\{\|x\|_n, \|y\|_m\} \quad , \quad \|\alpha x \beta\|_n \leq \|\alpha\| \|x\|_n \|\beta\|,$$

for $x \in M_n(X), y \in M_m(Y)$, and $\alpha, \beta \in M_n$. A *complete contraction* is a linear map $T : X \to Y$ satisfying

$$\|[T(x_{ij})]\|_n \leq \|[x_{ij}]\|_n,$$

for all $n \in \mathbb{N}, [x_{ij}] \in M_n(X)$. A *complete isometry* satisfies the same relation, with '$\leq$' replaced by '$=$'. Ruan's theorem [**9**] states that up to complete isometry, abstract operator spaces are the same thing as concrete operator spaces. Every Banach space X becomes an operator space in canonical ways. If X is an operator space then X^*, equipped with matrix norms coming from the identification $M_n(X^*) \cong CB(X, M_n)$, is called the *dual operator space* of X. See e.g. [**2**] for some basic properties of this duality.

For simplicity in this paper, we will only consider algebras which are *unital*, unless we say otherwise, and we will assume that the identity or unit has norm 1. By a *concrete operator algebra* we mean a closed subalgebra A of $B(H)$. For simplicity, and for purposes of comparison, we define a Banach algebra to be a unital algebra A which is also a Banach space, such that $\|ab\| \leq \|a\|\|b\|$ for $a, b \in B$. This suggests defining an *operator Banach algebra* to be a unital algebra which is also an operator space, such that $\|ab\|_n \leq \|a\|_n \|b\|_n$, for all $n \in \mathbb{N}$ and $a, b \in M_n(A)$. Here ab is the usual product of matrices. This looks innocent enough, a natural variant of the definition of a Banach algebra. But in fact:

THEOREM 1.1. (Blecher, Ruan and Sinclair) *Up to completely isometric homomorphism, operator Banach algebras are exactly the operator algebras.*

The question at hand is: are operator algebras Banach algebras, or are they operator Banach algebras? Of course they are both; but we want to know if the 'matrix norms' above are really necessary in the study of operator algebras. Of course this question depends largely on the particular applications or questions one has in mind. The situation in this regard for operator algebras is still being ironed out, and one of the purposes of Section 2 of this article is to emphasize some questions which might tip the balance here. The drawback of the operator space approach, for those unused to this subject, is that the family $\{\|\cdot\|_n\}_{n\geq 2}$ seems to be a heavy additional burden. In the authors experience, the matrix norms are rarely burdensome. Indeed usually if a result involving operator spaces can be proved for the 'first level' ($n = 1$), then the higher levels will follow in a routine way.

Although the definitions of Banach algebras and operator Banach algebras are so similar, and although Banach algebraic ideas and techniques always have, and always will, play a key role in the study of operator algebras, it is clear that the ensuing theories are for the most part divergent. Indeed one may argue that operator algebras are much more like C^*-algebras than they are like general Banach algebras (at least for many questions)[2]. If one likes, one may view operator algebras as the 'noncommutative function algebras'. By a function algebra' we mean

[2]To avoid a possible confusion, we remark in passing that there is another important 'operator space variant' of the definition of a Banach algebra, which is much more like general Banach algebras. Namely one may consider an algebra A which is also an operator space, such that $\|[a_{ij}b_{kl}]\|_n \leq \|a\|_n\|b\|_n$, for all $n \in \mathbb{N}$ and $a, b \in M_n(A)$. The i, k here index rows, and the j, l index columns. We mentioned these algebras in [**2**], and a few years later they were closely

here a subalgebra of a commutative C^*-algebras; and thus every function algebra is an operator algebra. Of course function algebras are often closer in spirit to $C(K)$ spaces than they are to general commutative Banach algebras, and a similar principle holds in the noncommutative case. For example, a main tool for studying operator algebras is Arveson's noncommutative version of the Shilov boundary [**1**], sometimes known as the *C^*-envelope*. For many more details about this and other topics concerning operator space aspects of operator algebras, better referencing of the literature, etc., we of course recommend [**4**].

2. Conditional expectations and the duality of operator algebras

Operator algebras have very many pleasant properties that general Banach algebras do not share. For example, in [**5**] we showed the following variant of an important theorem of Tomiyama on conditional expectations on C^*-algebras (see e.g. [**15**, Theorem III.3.4]):

THEOREM 2.1. *Let A be an operator algebra, and let $P : A \to A$ be a completely contractive projection (i.e. $P \circ P = P$) with $P(1) = 1$, and whose range is a subalgebra B of A. Then P is a 'conditional expectation', by which we mean that*

$$P(b_1 a b_2) = b_1 P(a) b_2$$

for all $a \in A, b_1, b_2 \in B$.

It can be seen easily that for function algebras, the map P in Theorem 2.1 is completely contractive if it is contractive. Indeed, this is true for any linear map into a subspace of a $C(K)$ space (this is an easy exercise, or see e.g. [**9**, Proposition 2.2.6]). Hence in this case Theorem 2.1 holds with the word 'completely' dropped. This is also known to be true for C^*-algebras (Tomiyama's result does not require complete contractivity). Thus one may ask:

QUESTION 1. For which classes of Banach algebras (or for which contractive projections P on a fixed Banach algebra) does Theorem 2.1 hold with the word 'completely' dropped? In particular, does it hold for the class of operator algebras?

The class of all Banach algebras certainly does not have the property in Question 1, as one may see by experimenting with three dimensional examples:

EXAMPLE. Let B be an algebra spanned by three idempotents $\{1, a, b\}$, where $ab = ba = a$. Defining $\|\lambda 1 + \mu a + \nu b\| = |\lambda| + |\mu| + |\nu|$, for $\lambda, \mu, \nu \in \mathbb{C}$, makes B a Banach algebra. However B does not have the above property, as one may see by taking $P(\lambda 1 + \mu a + \nu b) = \lambda 1 + \mu a$.

Unfortunately, one can show that there is no operator algebra norm on B for which P is contractive (we omit the elementary but tedious computation, which relies on the special structure of idempotent operators on a Hilbert space). However it seems likely to us that a clever variant of this example should yield an operator algebra without the property in Question 1.

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investigated by Ruan and coauthors. These algebras have been called *quantized Banach algebras* in [**9**], and are now extremely important in noncommutative harmonic analysis for example (see e.g. the articles of Spronk and Runde in the present volume). It is easy to see that every Banach algebra A is one of these algebras, for a canonical operator space structure on A. However these algebras are generally unrelated to operator algebras, at least from the perspective of the current paper, and hence they will not be mentioned again.

Another nice property operator algebras possess is *Arens regularity.* Indeed C^*-algebras have this property, and the property is hereditary. We refer the reader to any book on Banach algebras for definitions (see e.g. [**7**]); and to [**8**] for a multitude of details on Arens regularity and the Arens product.

A case of Question 1 that is of interest occurs when B is an Arens regular Banach algebra which is a dual Banach space. The canonical inclusion $B_* \to B^*$ dualizes to give a contractive projection Q from B^{**} onto the copy of B, viewed as a subalgebra of B^{**}. Clearly $Q(1) = 1$. Le Merdy studied this situation in [**13**]; and observed most of the following (see also the proof of [**5**, Corollary 5.2]):

PROPOSITION 2.2. *Let B, Q be as in the last paragraph. Then the product on B is separately weak* continuous if and only if Q is a 'conditional expectation' in the sense of 2.1, and if and only if Q is a homomorphism.*

In the light of this we obtain as a 'special case' of Question 1:

QUESTION 2. Which Arens regular Banach algebras, which are also dual spaces, necessarily have a separately weak* continuous product?

We imagine that some Banach algebraists must have encountered the last question, and have interesting examples at hand. Recently, Lau has communicated to us an example which he found together with Dales, of an Arens regular (unital) semigroup algebra $\ell^1(S)$ whose product is not separately weak* continuous (see also [**16**]). Indeed Lau has mentioned to us a useful criterion for when the product on $\ell^1(S)$ is not separately weak* continuous. In contrast, Question 2 has an affirmative answer for unital function algebras, by the remark above Question 1.

Godefroy and Iochum proved in [**10**] that any Banach algebra product on the second dual of a C^*-algebra A (we keep the same identity), is necessarily separately weak* continuous. It follows that there is a unique Banach algebra product on the second dual of A^{**} extending the product of A, namely the Arens product. Analogues of these facts remain true for operator Banach algebras:

THEOREM 2.3. *Let A be a unital operator algebra.*

(1) *If A is a dual operator space then the product on A is necessarily separately weak* continuous.*
(2) *There is a unique operator Banach algebra product on A^{**} extending the product of A, namely the Arens product.*

Note that (1) follows from Proposition 2.2 and Theorem 2.1 (but was first proved in [**3**] using noncommutative M-ideal theory). Item (2) is immediate from (1), since if A is Arens regular then the Arens product is the unique separately weak* continuous product on A^{**}; this was noticed during conversations with Christian Le Merdy.

QUESTION 3. Does (1) of Theorem 2.3 remain true for nonunital operator algebras?

By facts in [**11**] this is equivalent to asking if every 'quasimultiplier' of a dual operator space X is separately weak* continuous. This looks like it should be an easy question, however note that a counterexample to this question implies that not every one-sided multiplier (in our sense of [**3, 6**]) on X is weak* continuous, and would solve in the negative some other questions left open in [**3**].

One may also ask if (1) of Theorem 2.3 remains true if one replaces 'dual operator space' by dual Banach space'. This is just Question 2 restricted to the class of operator algebras.

QUESTION 4. Does (2) of Theorem 2.3 remain true without the matrix norms? That is, is the Arens product the only Banach algebra product on A^{**} extending the product on an operator algebra A, with respect to which A^{**} is isometrically isomorphic to an operator algebra?

It would also be interesting to consider other Banach algebraic duality properties, such as those concerning topological centers (see e.g. [**12**]), in the category of operator algebras.

Related to the above results is Sakai's important characterization of von Neumann algebras. Sakai's theorem says that 'the von Neumann algebras are exactly the C^*-algebras which possess a Banach space predual'. The word 'exactly' here means 'via a weak* homeomorphic $*$-isomorphism π'. When considering the nonselfadjoint operator algebra situation, one sees that there could be several variants of such a theorem, corresponding to whether one wants to keep 'Banach space predual' or replace it with 'operator space predual'; and whether π should be a weak* homeomorphic isometric isomorphism, or a weak* homeomorphic completely isometric one.

THEOREM 2.4. (Le Merdy [**13, 14**]) *Let A be a (possibly nonunital) operator algebra with an operator space predual. Then the product on A is separately weak* continuous if and only if A is completely isometrically isomorphic, via a weak* homeomorphic homomorphism, to a weak* closed subalgebra of $B(H)$, for some Hilbert space H.*

We call an operator algebra satisfying the equivalent conditions in Le Merdy's theorem a *dual operator algebra.* Combining Theorem 2.3 (1) and Theorem 2.4 gives:

COROLLARY 2.5. *A unital operator algebra with an operator space predual is a dual operator algebra.*

Thus Question 3 may be restated: Is Corollary 2.5 true for nonunital operator algebras? Question 2, in the particular case of operator algebras, reduces to:

QUESTION 5. Is an operator algebra with a Banach space predual *isometrically* isomorphic, via a weak* continuous isomorphism, to a weak* closed subalgebra of $B(H)$?

We were able to show in [**3**] that the answer to Question 5 is negative in general if 'isometrically' is replaced by 'completely isometrically'.

Finally, we turn to the 'commutative case' of function algebras. As we said at the end of the introduction, every function algebra is an operator algebra. By [**2**, Corollary 2.8], it is easy to see that a function algebra with a predual Banach space has a predual operator space. Thus, by the observation before Question 1, and by Theorem 2.4, the answer to Question 5 is in the affirmative for all unital function algebras. That is, every unital function algebra A with a Banach space predual, is isometrically isomorphic, via a weak* homeomorphism, to a weak* closed subalgebra of $B(H)$, for some Hilbert space H. That is, every such A is what is sometimes called a 'uniform dual algebra' in the function algebra literature (see e.g.

[**17**]; we remark that the assertion in the last line answers a question posed in that paper). However the following question (dating at least as far back as [**17**]) is still open:

QUESTION 6. Is every unital function algebra possessing a Banach space predual, isometrically isomorphic, via a weak* continuous isomorphism, to a weak* closed subalgebra of an L^∞-space (that is, of a commutative von Neumann algebra)?

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Jordan Banach algebras in harmonic analysis

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ABSTRACT. We expose some useful connections between Jordan Banach algebras and harmonic analysis on Riemannian symmetric spaces, and prove some new results in this context.

1. Jordan Banach algebras

The concept of a Jordan algebra was introduced by Jordan, von Neumann and Wigner in [**14**], with the aim to formulate an algebraic model for quantum mechanics. They introduced the notion of an *r-number system* which is, in present day terminology, a *finite-dimensional, formally real, Jordan algebra*, and they classified these algebras completely. Actually, the term "*Jordan algebra*" first appeared in a paper by Albert [**1**]. It denotes an algebra of linear transformations closed in the product $A \circ B = 1/2(AB + BA)$. The study of such an algebra was initiated by P. Jordan [**13**] earlier, in connection with quantum physics, which was seminal in the development of the theory of r-number systems.

In what follows, we discuss Jordan algebras with geometric overtones and show in this section when a non-surjective linear isometry preserves Jordan structures. We explain in the next section the close connections between Jordan Banach algebras and symmetric manifolds. In the last section, we show that the bounded harmonic functions on a Riemannian symmetric space form an associative Jordan algebra, and that amenability is necessary for the absence of a non-constant bounded harmonic function on a Riemannian symmetric space.

A *Jordan algebra* is a commutative, but not necessarily associative, algebra satisfying the *Jordan identity*

$$(ab)a^2 = a(ba^2).$$

We restrict our attention to only real or complex algebras. A real or complex Banach space $\mathcal{A}$ is called a *Jordan Banach algebra* if it is a Jordan algebra and the norm satisfies

$$\|ab\| \leq \|a\|\|b\| \qquad (a, b \in \mathcal{A}).$$

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Every Banach algebra $\mathcal{A}$ is a Jordan Banach algebra in the following Jordan product:

$$a \circ b = \frac{1}{2}(ab + ba) \qquad (a, b \in \mathcal{A})$$

where the product on the right is the original product. Therefore one can view Jordan Banach algebras as a non-associative generalization of (commutative) Banach algebras.

On a Jordan algebra $\mathcal{A}$, one defines the *Jordan triple product* by

$$\{a, b, c\} = (ab)c + (cb)a - (ac)b.$$

Each $x \in \mathcal{A}$ induces a *quadratic* operator $Q(x) : \mathcal{A} \longrightarrow \mathcal{A}$, given by

$$Q(x)(y) = \{x, y, x\}.$$

We define the multiplication operator $L(x) : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$L(x)(y) = xy.$$

Let $\mathcal{A}$ have an identity e. An element $x \in \mathcal{A}$ is said to be *invertible* if there exists $x^{-1} \in \mathcal{A}$ such that $xx^{-1} = e$ and $x^2x^{-1} = x$. This is equivalent to saying that the map $Q(x)$ is invertible with inverse $Q(x)^{-1} = Q(x^{-1})$.

A real Jordan algebra is called *formally real* if $\sum_j a_j^2 = 0$ implies all $a_j = 0$. A real Jordan Banach algebra $\mathcal{A}$ is called a *JB-algebra* if it satisfies

$$\|a^2\| = \|a\|^2$$
$$\|a^2\| \leq \|a^2 + b^2\|$$

for all $a, b \in \mathcal{A}$. The finite-dimensional unital formally real Jordan algebras are exactly the finite-dimensional JB-algebras [**10**]. The complex version of JB-algebras are the JB*-algebras. A complex Jordan Banach algebra $\mathcal{A}$ is called a *JB*-algebra* if it has an involution $*$ such that, for all $a \in \mathcal{A}$,

$$\|a^*\| = \|a\|,$$
$$\|a^{(3)}\| = \|a\|^3$$

where we define $a^{(3)} = \{a, a^*, a\}$ and include the involution in the Jordan triple product: $\{a, b^*, c\}$. It has been shown in [**22**] that the JB*-algebras are precisely the complexifications of JB-algebras. Every C*-algebra $\mathcal{A}$ is a JB*-algebra in the Jordan product

$$a \circ b = \frac{1}{2}(ab + ba) \qquad (a, b \in \mathcal{A})$$

and $\mathcal{A}$ is abelian if, and only if, $(\mathcal{A}, \circ)$ is an associative JB*-algebra. One can regard JB*-algebras as a non-associative generalization of C*-algebras and indeed, JB*-algebras are also called *Jordan C*-algebras.* However, historically many important aspects of Jordan theory were developed in the study of symmetric manifolds in differential geometry as one will have a glimpse of this later.

We note that the norm on a JB*-algebra is unique, in fact, a Jordan product isomorphism between two JB*-algebras is necessarily isometric. Conversely, a *surjective* linear isometry φ between two JB*-algebras preserves the Jordan *triple* product:

$$\varphi\{a, b^*, c\} = \{\varphi(a), \varphi(b)^*, \varphi(c)\}$$

and if the algebras as well as φ are unital, then φ preserves the Jordan product. This result is a special case of a well-known result of Kaup [**16**] for the larger class of JB*-triples, proved elegantly by complex geometric method (see also [**5, 11**]). It subsumes Kadison's result [**15**] for surjective linear isometries of C*-algebras. The result is false if φ is not surjective. We now refine on Kaup's result to include the non-surjective case. The following two new results are actually valid for the larger class of JB*-triples although we suppress the discussion of JB*-triples in this paper.

We first recall that a map $h : D \longrightarrow U$ between open sets in complex Banach spaces Z and W, respectively, is called *holomorphic* if the Fréchet derivative $h'(a) : Z \longrightarrow W$ exists for every $a \in D$, where $h'(a)$ is a complex linear map satisfying

$$\lim_{t\to 0} \frac{\|h(a+t) - h(a) - h'(a)(t)\|}{\|t\|} = 0.$$

A holomorphic map $h : D \longrightarrow U$ is called *biholomorphic* if it is bijective and the inverse h^{-1} is also holomorphic. The open unit ball of a Banach space Z will be denoted by Z_0. Let Aut Z_0 be the automorphism group of Z_0, consisting of all biholomorphic maps from Z_0 onto itself. For a JB*-algebra $\mathcal{A}$, the basic elements in Aut $\mathcal{A}_0$ are the Möbius transformations. To describe them, we first define the following two fundamental linear operators on a JB*-algebra $\mathcal{A}$. For $x, y \in \mathcal{A}$, the *box operator* $x\Box y : \mathcal{A} \longrightarrow \mathcal{A}$ and the *Bergman operator* $B(x,y) : \mathcal{A} \longrightarrow \mathcal{A}$ are defined by

$$\begin{aligned} (x\Box y)(z) &= \{x, y^*, z\}, \\ B(x,y)(z) &= z - 2\{x, y^*, z\} + \{x, \{y, z^*, y\}^*, x\}. \end{aligned}$$

Given $a \in \mathcal{A}_0$, we define the *Möbius transformation of* $\mathcal{A}_0$, *induced by* a, to be the biholomorphic map $h_a : \mathcal{A}_0 \longrightarrow \mathcal{A}_0$ given by

$$h_a(z) = a + B(a,a)^{1/2}(I + z\Box a)^{-1}(z) \qquad (z \in \mathcal{A})$$

where I is the identity operator. We have $h_a(0) = a$, $h_a(-a) = 0$ and $h_a^{-1} = h_{-a}$. We also have the Fréchet derivatives $h'_a(0) = B(a,a)^{1/2}$ and $h'_{-a}(a) = B(a,a)^{-1/2}$ (cf. [**16**]).

If $\mathcal{A}$ is a C*-algebra, we have the following formula for the Möbius transformation, due to Potapov [**19**] and Harris [**11**]:

$$h_a(z) = (1 - aa^*)^{-1/2}(a + z)(1 + a^*z)^{-1}(1 - a^*a)^{1/2}$$

which reduces to the familiar Möbius transformation on the complex open unit disc if $\mathcal{A} = \mathbb{C}$.

LEMMA 1.1. *Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear isometry, not necessarily surjective, between JB*-algebras $\mathcal{A}$ and $\mathcal{B}$. Let $a \in \mathcal{A}_0$ and let $f \in$ Aut $\varphi(\mathcal{A})_0$ be such that $f(\varphi(a)) = 0$. Then*

$$f(0) = -f'(\varphi(a))(\varphi(B(a,a)^{1/2}(a))).$$

PROOF. Let $h = f\varphi h_a : \mathcal{A}_0 \longrightarrow \varphi(\mathcal{A})_0$. Then h is biholomorphic and $h(0) = 0$. Hence h is linear by Cartan's uniqueness theorem [**20**, p. 215] and on $\mathcal{A}_0$, we have $h = h'(0) = (f\varphi h_a)'(0) = (f\varphi)'(h_a(0))h'_a(0) = (f\varphi)'(a)B(a,a)^{1/2}$. Evaluating h at $-a$, we get the formula. $\Box$

We note that $\varphi h_{-a}\varphi^{-1}$ is an automorphism of $\varphi(\mathcal{A})_0$ and maps $\varphi(a)$ to 0. For a C*-algebra, we have $B(a,a)^{1/2}(a) = (1 - aa^*)^{1/2}a(1 - a^*a)^{1/2} = a - aa^*a$ since $(1 - aa^*)^{1/2}a = a(1 - a^*a)^{1/2}$. Therefore we have $B(a,a)^{1/2}(a) = a - \{a, a^*, a\}$ for a JB*-algebra, by considering the JB*-subalgebra generated by a and a^* which is a Jordan subalgebra of a C*-algebra by the Shirshov-Cohn theorem [**10**, 2.4.14].

Since $B(a,a) = B(-a,-a)$, we have $h_{-a}(-z) = -h_a(z)$. It follows that, if $D = -D$ is a subset of the open unit ball of a JB*-algebra, invariant under h_a, then it is also invariant under h_{-a} and $h_a(D) = D$.

The following refinement of Kaup's result in [**16**, Proposition 5.5] generalizes as well as giving a geometric proof of Kadison's result for C*-algebras [**15**].

THEOREM 1.2. *Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear isometry, not necessarily surjective, between JB*-algebras $\mathcal{A}$ and $\mathcal{B}$. Let $a \in \mathcal{A}_0$ and let $h_{\varphi(a)} : \mathcal{B}_0 \longrightarrow \mathcal{B}_0$ be the Möbius transformation induced by $\varphi(a)$. If $h_{\varphi(a)}(\varphi(\mathcal{A})_0) \subset \varphi(\mathcal{A})_0$, then we have*

$$\varphi\{a, a^*, a\} = \{\varphi(a), \varphi(a)^*, \varphi(a)\}.$$

In particular, if φ is surjective, then φ is a Jordan triple isomorphism.

PROOF. Let f be the restriction to of the Möbius transformation $h_{-\varphi(a)} \in \operatorname{Aut} \mathcal{B}_0$ to $\varphi(\mathcal{A})_0$. Then $f \in \operatorname{Aut} \varphi(\mathcal{A})_0$, $f(\varphi(a)) = 0$ and the derivative $f'(\varphi(a)) : \varphi(\mathcal{A}) \longrightarrow \varphi(\mathcal{A})$ is the restriction of the derivative $h'_{-\varphi(a)}(\varphi(a)) : \mathcal{B} \longrightarrow \mathcal{B}$ which is equal to $B(\varphi(a), \varphi(a))^{-1/2}$. By Lemma 1.1, we have

$$-\varphi(a) = f(0) = -f'(\varphi(a))(\varphi(B(a,a)^{1/2}(a)) = -B(\varphi(a), \varphi(a))^{-1/2}\varphi(a - a^{(3)}).$$

It follows that $\varphi(a) - \varphi(a)^{(3)} = B(\varphi(a), \varphi(a))^{1/2}(\varphi(a)) = \varphi(a) - \varphi(a^{(3)})$ which gives $\varphi(a)^{(3)} = \varphi(a^{(3)})$.

Finally, if φ is surjective, then $\varphi(\mathcal{A})_0 = \mathcal{B}_0$ and φ preserves the triple product by the polarization identity

$$\{a, b^*, c\} \quad = \quad \frac{1}{8} \sum_{\alpha^4 = \beta^2 = 1} \alpha\beta\{(a + \alpha b + \beta c), (a + \alpha b + \beta c)^*, (a + \alpha b + \beta c)\}.$$

$\square$

2. Riemannian symmetric spaces

A *Riemannian symmetric space* X is a connected Riemannian manifold in which every point is an isolated fixed-point of an involutive isometry of X. The Euclidean space $\mathbb{R}^d$ is such a space with the involutive isometry φ_p at each point $p \in \mathbb{R}^d$ given by $\varphi_p(x) = 2p - x$. A Riemannian symmetric space can be represented as the coset space G/H of a Lie group G by a maximal compact subgroup H. An important class of examples are the symmetric cones about which full details of what follows can be found in [**6**]. A classic reference for Riemannian symmetric spaces is [**12**].

Let V be a finite-dimensional real inner product space with inner product $\langle \cdot, \cdot \rangle$. An open cone $\Omega \subset V$ is called *symmetric* if it satisfies

(i) (self-duality) $\Omega = \{v \in V : \langle v, x \rangle > 0 \ \forall x \in \overline{\Omega} \backslash \{0\}\}$;
(ii) (homogeneity) given $x, y \in \Omega$, there is a linear isomorphism $h : V \longrightarrow V$ such that $h(\Omega) = \Omega$ and $h(x) = y$.

EXAMPLE 2.1. For $n > 2$, the *Lorentz cone* $\Lambda_n \subset \mathbb{R}^n$, defined below, is symmetric.

$$\Lambda_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 > 0 \text{ and } x_1^2 - x_2^2 - \cdots - x_n^2 > 0\}.$$

EXAMPLE 2.2. Let V be the linear space of all real symmetric $m \times m$ matrices, equipped with the inner product

$$\langle A, B \rangle = \operatorname{trace}(AB).$$

The cone Ω consisting of positive definite symmetric $m \times m$ real matrices is symmetric in V.

EXAMPLE 2.3. Let $\mathcal{A}$ be a finite-dimensional unital formally real Jordan algebra. The interior Ω of the cone $\{a^2 : a \in \mathcal{A}\}$ is a symmetric cone with respect to the inner product $\langle x, y\rangle = \operatorname{trace} L(xy)$. The elements in Ω are invertible and $Q(b)Q(a^{-1})$ is a linear automorphism of $\mathcal{A}$, sending Ω to itself, and a^2 to b^2.

Every symmetric cone Ω can be decomposed as a sum of irreducible ones:

$$\Omega = \Omega_1 + \cdots + \Omega_k$$

where each Ω_i is a symmetric cone in a subspace V_i of V and cannot be decomposed further into a sum of symmetric cones.

A symmetric cone Ω in V can be given a Riemannian metric g which makes it into a Riemannian symmetric space. To do this, we define the *characteristic function* $\varphi : \Omega \longrightarrow \mathbb{R}$ by

$$\varphi(x) = \int_\Omega \exp(-\langle x, y\rangle)dy$$

where dy denotes the Euclidean measure on V. For each $v \in V$, we have the directional derivative

$$D_v \log \varphi(x) = \frac{d}{dt}\bigg|_{t=0} \log \varphi(x + tv).$$

One can define a Riemannian metric g on Ω by

$$g_x(u, v) = D_u D_v \log \varphi(x) \qquad (x \in \Omega, u, v \in V). \tag{2.1}$$

In fact, this metric can be constructed equivalently via a Jordan algebra thereby algebraic methods can be applied to analysis on symmetric cones.

We now describe the seminal result of Koecher [**17**] and Vinberg [**21**] which links Jordan algebras to geometry and analysis.

THEOREM 2.4 (Koecher and Vinberg). *Let Ω be a symmetric cone in a real inner product space V. Then V can be equipped with a formally real Jordan product and an identity e such that Ω is the interior of its closure which is given by*

$$\overline{\Omega} = \{x^2 : x \in V\}.$$

Further, Ω is the connected component of e in the space of invertible elements in V.

Using solely the above Jordan structure of V, one can define the following equivalent Riemannian metric on Ω:

$$\gamma_x(u, v) = \operatorname{trace} L(\{x^{-1}, u, x^{-1}\}v) \qquad (x \in \Omega,\ u, v \in V).$$

If Ω is irreducible, then the Riemannian metric g in (2.1) is a scalar multiple of γ.

Let $\{e_i\}$ be an orthonormal basis of V with respect to the inner product $\langle u, v\rangle = \operatorname{trace} L(uv)$ and let

$$\gamma_{ij}(x) = \gamma_x(e_i, e_j).$$

Denote the inverse of the matrix $\gamma = (\gamma_{ij})$ by (γ^{ij}). The Laplace-Beltrami operator on Ω now has the form

$$\Delta = \frac{1}{\sqrt{\det \gamma}} \sum_{i,j} \frac{\partial}{\partial x_i}\left(\sqrt{\det \gamma}\, \gamma^{ij} \frac{\partial}{\partial x_j}\right)$$

which will be discussed in a wider context in the next section.

We have seen that the symmetric cones are exactly the interiors of the cones of finite-dimensional JB-algebras. The important connection of JB*-algebras with geometry can be seen from the following result of Braun, Kaup and Upmeier [**2**].

THEOREM 2.5. *A bounded domain in a complex Banach space is a symmetric tube domain if, and only if, it is biholomorphic to the open unit ball of a unital JB*-algebra.*

A bounded domain in a complex Banach space is called a *symmetric tube domain* if it is biholomorphic to a domain of the form

$$V + i\Omega := \{a + ib : a \in V, b \in \Omega\}$$

which is contained in the complexification $V + iV$ of a real Banach space V, where Ω is an open cone in V and every point in $V + i\Omega$ is an isolated fixed-point of a holomorphic involution $s : V + i\Omega \longrightarrow V + i\Omega$. If $V = \mathbb{R}$ and $\Omega = (0, \infty)$, then $V + i\Omega = \mathbb{R} + i(0, \infty)$ is just the upper half plane in $\mathbb{C}$. Therefore one can view, from the above theorem, the open unit balls of JB*-algebras as generalizations of the upper half complex plane.

3. Harmonic functions

In this section, we show that the bounded harmonic functions on a Riemannian symmetric space form an associative JB-algebra. We also give a necessary condition for a Liouville theorem for these spaces. We first note that a Riemannian symmetric space Ω is a complete manifold and if Ω has non-negative sectional curvature, then the bounded harmonic functions on Ω are constant. Indeed, Yau [**23**] has shown that on a complete Riemannian manifold of non-negative Ricci curvature, every bounded harmonic function is constant. Therefore we need only consider Riemannian symmetric spaces with negative sectional curvature. *Let X be such a manifold throughout this section.* The identity component G of the isometry group of X is a semisimple Lie group acting transitively on X which can then be represented naturally as a left coset space G/H, where H is the isotropy group at any point of X and is a maximal compact subgroup of G [**12**]. Let Δ be the Laplace-Beltrami operator on X. This operator is defined for symmetric cones in the previous section. Following [**8**, Definition 5], we define a *Laplace operator* $\mathcal{L}$ on X to be any elliptic second-order differential operator (left) invariant under G and annihilating constants. A real function $f \in C^\infty(X)$ is *harmonic* if $\Delta f = 0$, and is called *strongly harmonic* if $\mathcal{L}f = 0$ for any Laplace operator $\mathcal{L}$ on X.

Let σ be a probability measure on G. A Borel function $f : X \longrightarrow \mathbb{R}$ is called σ*-harmonic* if the *inverted* function $\widetilde{f}(gH) := f(g^{-1}H)$ satisfies the convolution equation

$$\widetilde{f}(x) = \sigma * \widetilde{f}(x) := \int_G \widetilde{f}(g^{-1}.x)d\sigma(g) \qquad (x \in X)$$

where $x \in X \mapsto g.x \in X$ $(g \in G)$ is the natural action of G on $X = G/H$. A measure σ on G is called *H-invariant* if $d\sigma(hg) = d\sigma(gh) = d\sigma(g)$ for any $h \in H$. We note that, if σ is absolutely continuous, then the bounded σ-harmonic functions are continuous. The following well-known result is due to Furstenberg [**8**].

THEOREM 3.1. *Let $X = G/H$ be the above manifold. Then there is an H-invariant absolutely continuous probability measure σ on G such that the following conditions are equivalent for a bounded function function f on X:*

(i) *f is harmonic;*
(ii) *f is strongly harmonic;*
(iii) *f is σ-harmonic.*

Although the pointwise product of two harmonic functions need not be harmonic, we show that the space $H(X)$ of bounded harmonic functions on X forms an associative JB-algebra in certain product, using the following device.

LEMMA 3.2. *Let $L^\infty(X)$ be the real Banach algebra of essentially bounded functions on X, with respect to a (left) G-invariant Radon measure on X. Let σ be a probability measure on G. Then there is a contractive projection $P : L^\infty(X) \longrightarrow L^\infty(X)$ whose range is*

$$H_\sigma(X) := \{f \in L^\infty(X) : f = \sigma * f\}.$$

PROOF. The construction is similar to that in [**4**, Proposition 2.2.5]. Define a weak* continuous linear map $\Lambda : L^\infty(X) \longrightarrow L^\infty(X)$ by

$$\Lambda(f) = \sigma * f.$$

Then Λ is a contraction since σ is a probability measure. Let $L^\infty(X)^{L^\infty(X)}$ be equipped with the product weak* topology $\mathcal{T}$ and let K be the $\mathcal{T}$-closed convex hull of $\{\Lambda^n : n = 1, 2, \ldots\}$ in this space, where $\Lambda^n = \Lambda \circ \cdots \circ \Lambda$ (n-times). Define an affine $\mathcal{T}$-continuous map $\Phi : K \longrightarrow K$ by

$$\Phi(\Gamma)(f) = \sigma * \Gamma(f) \qquad (\Gamma \in K, f \in L^\infty(X)).$$

By the Markov-Kakutani fixed-point theorem, there exists $P \in K$ such that $\Phi(P) = P$ and P is the required contractive projection. □

PROPOSITION 3.3. *Let Ω be a Riemannian symmetric space. Then the space $H(\Omega)$ of bounded harmonic functions on Ω is linearly isometric to a real abelian C^*-algebra.*

PROOF. We need only consider the case $\Omega = X = G/H$ as above. Let σ be the H-invariant absolutely continuous probability measure in Theorem 3.1. By the above lemma, $H_\sigma(X)$ is the range of a contraction projection on $L^\infty(X)$. Since $H_\sigma(X)$ contains constant functions, a result in [**18**, p.343] implies that it is isometric to the space of real continuous functions on a compact Hausdorff space (see also [**7**]). Hence the result follows as the map $f \in H(X) \mapsto \widetilde{f} \in H_\sigma(X)$ is a linear isometry. □

We conclude with the following necessary condition for the Liouville Theorem. A left coset space $X = G/H$ is called *G-amenable* (or simply, *amenable*) if G acts amenably on X in the sense of [**9**], that is, if there is a left-invariant mean on $L^\infty(X)$.

THEOREM 3.4. *Let $X = G/H$ be a Riemannian symmetric space with negative sectional curvature. If every bounded harmonic function on X is constant, then X is amenable.*

PROOF. Since X has negative sectional curvature, we can apply Furstenberg's result in Theorem 3.1. Let σ be the H-invariant absolutely continuous probability measure in Theorem 3.1 such that $H(X) = \{f \in L^\infty(X) : \widetilde{f} = \sigma * \widetilde{f}\}$. By assumption, $H(X)$ consists of constant functions. Let $H_\sigma(G)$ be the space of all bounded Borel functions $F : G \longrightarrow \mathbb{R}$ satisfying the convolution equation

$$F(g) = \sigma * F(g) := \int_G F(y^{-1}g)d\sigma(y) \qquad (g \in G).$$

We show that $H_\sigma(G)$ consists of only constant functions. Let $F \in H_\sigma(G)$ and define a function $f : X \longrightarrow \mathbb{R}$ by

$$f(gH) = \int_H F(gh)dh$$

where dh is the normalized Haar measure on the compact group H. Then $\widetilde{f}$ is harmonic on X since

$$\begin{aligned}(\sigma * f)(gH) &= \int_G f(y^{-1}gH)d\sigma(y) \\ &= \int_H \int_G F(y^{-1}gh)d\sigma(y)dh \\ &= \int_H F(gh)dh = f(gH).\end{aligned}$$

Hence $\widetilde{f} \in H(X)$ and is constant. So f is constant which gives

$$\int_H F(ah)dh = \int_H F(bh)dh \qquad (a, b \in G). \tag{3.1}$$

Using (3.1) and the H-invariance of σ, we have

$$\begin{aligned}F(e) &= \int_G F(y^{-1})d\sigma(y) = \int_G \int_H F(y^{-1})dhd\sigma(y) \\ &= \int_G \int_H F(y^{-1}h)dhd\sigma(y) = \int_G \int_H F(bh)dhd\sigma(y) \\ &= \int_H F(bh)dh\end{aligned}$$

for all $b \in G$. Apply the above to the right translation $F(\cdot a) \in H_\sigma(G)$, we obtain

$$F(a) = \int_H F(bha)dh \qquad (a, b \in G).$$

It follows that

$$\widetilde{F}(a) = \int_H \widetilde{F}(ahb)dh \qquad (a, b \in G) \tag{3.2}$$

where $\widetilde{F}(g) = F(g^{-1})$. The H-invariance of σ also implies that, for all $h \in H$ and $a \in G$,

$$\begin{aligned}F(ha) &= \int_G F(y^{-1}ha)d\sigma(y) \\ &= \int_G F(y^{-1}a)d\sigma(y) = F(a).\end{aligned}$$

Therefore we can define a function $\varphi : X \longrightarrow \mathbb{R}$ by

$$\varphi(aH) = \widetilde{F}(a).$$

By (3.2) and [**8**, Theorem 5], the function φ is harmonic. Hence φ is constant, that is, $\widetilde{F}$ is constant. So $H_\sigma(G)$ contains only constant functions and by [**4**, Corollary 2.2.8], G is amenable, and so is G/H. □

It follows from the above result that there are non-constant bounded harmonic functions on the non-amenable symmetric spaces $SL(n,\mathbb{R})/SO(n)$, where $SL(2,\mathbb{R})/SO(2)$ is the upper half plane.

Without differential structures, one can consider σ-harmonic functions on coset spaces G/H of locally compact groups G where H need not be compact and σ need not be H-invariant. It would be of interest to extend Theorem 3.4 to this setting. We note that amenability of G/H does not imply that of G, unless H is compact [**9**, Theorem 3.1].

Matrix-valued σ-harmonic functions on locally compact groups have been studied in [**3**]. As an analogue of σ-harmonic functions, the concept of a *harmonic functional* on the Fourier algebra $A(G)$ of a locally compact group G has been introduced in [**4**] and it has been shown that these harmonic functionals form a JB*-algebra [**4**, Proposition 3.3.5].

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