# Von Neumann algebraic $H^p$ theory

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Abstract. Around 1967, Arveson invented a striking noncommutative generalization of classical $H^\infty$, known as *subdiagonal algebras*, which include a wide array of examples of interest to operator theorists. Their theory extends that of the generalized $H^p$ spaces for function algebras from the 1960s, in an extremely remarkable, complete, and literal fashion, but for reasons that are 'von Neumann algebraic'. Most of the present paper consists of a survey of our work on Arveson's algebras, and the attendant $H^p$ theory, explaining some of the main ideas in their proofs, and including some improvements and short-cuts. The newest results utilize new variants of the noncommutative Szegö theorem for $L^p(M)$, to generalize many of the classical results concerning outer functions, to the noncommutative $H^p$ context. In doing so we solve several of the old open problems in the subject.

## 1. Introduction

What does one get if one combines the theory of von Neumann algebras, and that of Hardy's $H^p$ spaces? In the 1960s, Arveson suggested one way in which this could be done [**1, 2**], via his introduction of the notion of a *subdiagonal* subalgebra $A$ of a von Neumann algebra $M$. In the case that $M$ has a finite trace (defined below), $H^p$ may be defined to be the closure of $A$ in the noncommutative $L^p$ space $L^p(M)$, and our article concerns the 'generalization' of Hardy space results to this setting. In the case that $A = M$, the $H^p$ theory collapses to noncommutative $L^p$ space theory. At the other extreme, if $A$ contains no selfadjoint elements except scalar multiples of the identity, the $H^p$ theory will in the setting where $M$ is commutative, collapse to the classical theory of $H^p$-spaces associated to the so-called 'weak* Dirichlet algebras'—a class of abstract function algebras. Thus Arveson's setting formally merges noncommutative $L^p$ spaces, and the classical theory of $H^p$-spaces for abstract function algebras. We say more about the latter: around the early 1960's, it became apparent that many famous theorems about the classical $H^\infty$ space of bounded analytic functions on the disk, could be generalized to the setting of abstract function algebras. This was the work of several notable researchers, and

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in particular Helson and Lowdenslager [**21**], and Hoffman [**25**]. The paper [**55**] of Srinivasan and Wang, from the middle of the 1960s decade, organized and summarized much of this 'commutative generalized $H^p$-theory'. In the last few years, we have shown that all of the results from this classical survey, and essentially everything relevant in Hoffman's landmark paper [**25**], extend in a particularly literal way to the noncommutative setting of Arveson's subdiagonal subalgebras (en route solving what seemed to be the major open problems in the latter subject). Indeed, as an example of what some might call 'mathematical quantization', or noncommutative (operator algebraic) generalization of a classical theory, the program succeeds to a degree of 'faithfulness to the original' which seems to be quite rare. This in turn suggests that the analytic principles encapsulated in the classical theory are far more algebraic in nature than was even anticipated in the 1960s.

Since the classical theories mentioned above are quite beautiful, and since the noncommutative variant is such a natural place of application of von Neumann algebra and noncommutative $L^p$ space theory, this has been a very pleasurable labor, which we are grateful to have participated in. It seems timely to survey the main parts of our work, aiming this at a general audience, and including a description of a few of the von Neumann algebraic and noncommutative $L^p$ space methods that are needed. We also include improved routes through some of our proofs. There are two sections describing very recent results from [**10**]. For example, we give new variants of the noncommutative Szegö's theorem for $L^p(M)$, and using these, we generalize many of the classical results concerning outer functions to the noncommutative $H^p$ context. In particular, we develop in Section 7 the theory of outer functions in the simple and tidy 'antisymmetric algebra' case, proving almost all of our assertions here. Outers for more general algebras are treated in [**10**].

We write $H^\infty(\mathbb{D})$ for the algebra of analytic and bounded functions on the open unit disc $\mathbb{D}$ in the complex plane. As we just mentioned, and now describe briefly, around 1960 many notable mathematicians attempted to generalize the theory of $H^p$ spaces. The setting for their 'generalized $H^p$ function theory' was the following: Let $X$ be a probability space, and let $A$ be a closed unital-subalgebra of $L^\infty(X)$, such that:

$$\int fg = \int f \int g, \qquad f, g \in A. \tag{1.1}$$

The latter was the crucial condition which they had isolated as underpinning the classical generalized function theory. Note that this clearly holds in the case of $H^\infty(\mathbb{D}) \subset L^\infty(\mathbb{T})$, the integral being the normalised Lebesgue integral for the unit circle $\mathbb{T}$. We will suppose that $A$ is weak* closed (otherwise it may be replaced by its weak* closure). Write $[\mathcal{S}]_p$ for the closure of a set $\mathcal{S} \subset L^p$ in the $p$-norm, and define $H^p = [A]_p$ whenever $A$ satisfies any/all of the conditions described in the following theorem. Let $A_0 = \{f \in A : \int f = 0\}$. Combining fundamental ideas of many researchers, one may then prove [**25, 55, 26**] that:

Theorem 1.1. *For such $A$, the following eight conditions are equivalent:*

(i) *The weak* closure of $A + \bar{A}$ is all of $L^\infty(X)$.*
(ii) *The 'unique normal state extension property' holds, that is: if $g \in L^1(X)$ is nonnegative with $\int fg = \int f$ for all $f \in A$, then $g = 1$ a.e..*
(iii) *$A$ has 'factorization', that is: if $b \in L^\infty(X)$, then $b \geq 0$ and is bounded away from 0 iff $b = |a|^2$ for an invertible $a \in A$.*

(iv) *$A$ is 'logmodular', that is: if $b \in L^\infty(X)$ with $b$ bounded away from 0 and $b \geq 0$, then $b$ is a uniform limit of terms of the form $|a|^2$ for an invertible $a \in A$).*

(v) *$A$ satisfies the $L^2$-distance formula in Szegö's theorem, that is:* $\exp \int \log g = \inf\{\int |1-f|^2 g : f \in A, \int f = 0\}$, *for any nonnegative $g \in L^1(X)$.*

(vi) *Beurling invariant subspace property: every $A$-invariant closed subspace of $L^2(X)$ such that $[A_0 K]_2 \neq K$ is of the form $uH^2$ for a unimodular function $u$.*

(vii) *Beurling-Nevanlinna factorization property, that is: every $f \in L^2(X)$ with $\int \log|f| > -\infty$ has an 'inner-outer factorization' $f = uh$, with $u$ unimodular and $h \in H^2$ outer (that is, such that $1 \in [hA]_2$).*

(viii) *Gleason-Whitney property: there is a unique Hahn-Banach extension to $L^\infty(X)$ of any weak* continuous functional on $A$, and this extension is weak* continuous.*

It is worth remarking that almost none of the implications here are clear; indeed the theorem above constitutes a resumé of a network connecting several topics of great interest. The objects characterized here, the *weak* Dirichlet algebras*, are the topic of [**55**]. The theory goes on to show that they satisfy many other properties that generalize those of bounded analytic functions on the disk; e.g. $H^p$ variants of properties (vi) and (vii) for all $1 \leq p \leq \infty$, Jensen's inequality, the F & M Riesz theorem, Riesz factorization, the characterization of outer functions in terms of $\int \log f$, the criteria for inner-outer factorization, and so on.

For us, the primary question is the extent to which all of this survives the passage to noncommutativity. The remarkable answer is that in the setting of Arveson's subdiagonal subalgebras of von Neumann algebras, essentially everything does. A key point when trying to generalize to a von Neumann algebra framework, is that one must avoid most classical arguments which involve exponentials of functions—since the exponential map behaves badly if the exponent is not a normal operator. Thus we avoided some of the later, more sophisticated, routes through the classical theory (see e.g. [**16**]), and went back to older more algebraic methods of Helson-Lowdenslager, Hoffman, and others. Being based primarily on Hilbert space methods, these more easily go noncommutative. Although the *statements* of the results which we obtain are essentially the same as in the commutative case, and although the proofs and techniques in the noncommutative case may often be modeled loosely on the 'commutative' arguments of the last-mentioned authors, they usually are much more sophisticated, requiring substantial input from the theory of von Neumann algebras and noncommutative $L^p$-spaces. Sometimes completely new proofs have had to be invented (Jensen's inequality is a good example of this).

We now review some of the definitions we shall use throughout. For a set $\mathcal{S}$, we write $\mathcal{S}_+$ for the set $\{x \in \mathcal{S} : x \geq 0\}$. The word 'normal' applied to linear mappings as usual means 'weak* continuous'. We assume throughout that $M$ is a von Neumann algebra possessing a faithful normal tracial state $\tau$. Here 'faithful' means that $\mathrm{Ker}(\tau) \cap M_+ = (0)$, and 'tracial' means that $\tau(xy) = \tau(yx)$ for all $x, y \in M$. The existence of such $\tau$ implies that $M$ is a so-called *finite* von Neumann algebra. One consequence of this, which we shall use a lot, is that if $x^*x = 1$ in $M$, then $xx^* = 1$ too. Indeed $0 = 1 - \tau(x^*x) = \tau(1 - xx^*)$, and so $1 - xx^* = 0$ because $\tau$ is faithful. Applying the above to the partial isometry in the polar decomposition of any $x \in M$, implies, in operator theoretic terms, that $x$ is onto iff $x$ is invertible

iff $x$ is bounded below. From this in turn it follows that for any $a, b \in M$, $ab$ will be invertible precisely when $a$ and $b$ are separately invertible.

A *tracial subalgebra* of $M$ is a weak* closed subalgebra $A$ of $M$ such that the (unique) trace preserving[1] conditional expectation $\Phi : M \to A \cap A^* \stackrel{def}{=} \mathcal{D}$ (guaranteed by [**56**, p. 332]) satisfies:

$$\Phi(a_1 a_2) = \Phi(a_1)\,\Phi(a_2), \quad a_1, a_2 \in A. \tag{1.2}$$

(Note that (1.2) is a variant of the crucial formula (1.1) underpinning the entire theory.) A *finite maximal subdiagonal algebra* is a tracial subalgebra of $M$ with $A + A^*$ weak* dense in $M$. For brevity we will usually drop the word 'finite maximal' below, and simply say 'subdiagonal algebra'. In the classical function algebra setting [**55**], one assumes that $\mathcal{D} = A \cap A^*$ is one dimensional, which forces $\Phi = \tau(\cdot)1$. If in our setting this is the case, then we say that $A$ is *antisymmetric*. It is worth remarking that the antisymmetric maximal subdiagonal subalgebras of commutative von Neumann algebras are precisely the (weak* closed) weak* Dirichlet algebras. The simplest example of a noncommutative maximal subdiagonal algebra is the upper triangular matrices $A$ in $M_n$. Here $\Phi$ is the expectation onto the main diagonal. There are much more interesting examples from free group von Neumann algebras, etc. See e.g. [**2, 61, 35, 31, 37**]; and in the next paragraph we will mention a couple of examples in a little more detail. In fact much of Arveson's extraordinary original paper develops a core of substantial examples of interest to operator theorists and operator algebraists; indeed his examples showed that his theory unified part of the existing theory of nonselfadjoint operator algebras. Note too that the dropping of the 'antisymmetric' condition above, gives the class of subdiagonal algebras a generality and scope much wider than that of weak* Dirichlet algebras. Thus, for example, $M$ itself is a maximal subdiagonal algebra (take $\Phi = Id$). It is also remarkable, therefore, that so much of the classical $H^p$ theory does extend to *all* maximal subdiagonal algebras. However the reader should not be surprised to find a few results here which do (at present) require restrictions on the size of $\mathcal{D}$.

To get a feeling for how subdiagonal subalgebras can arise, we take a paragraph to mention very briefly just two interesting examples. See the papers referred to in the last paragraph for more examples, or more details. The first of these examples is due to Arveson [**2**, Section 3.2]. Let $G$ be a countable discrete group with a linear ordering which is invariant under left multiplication, say. For example, any free group is known to have such an ordering (see e.g. [**43**]). This implies that $G = G_+ \cup G_-, G_+ \cap G_- = \{1\}$. The subalgebra generated by $G_+$ in the group von Neumann algebra of $G$, immediately gives a subdiagonal algebra. For a second example (see [**61, 35, 31**]), if $\alpha$ is any one-parameter group of $*$-automorphisms of a von Neumann algebra $M$ satisfying a certain ergodicity property (and in particular, all those arising in the Tomita-Takesaki theory), naturally gives rise to a subdiagonal algebra $A \subset M$, coming from those elements of $M$ whose 'spectrum with respect to $\alpha$' lies in the nonnegative part of the real line.

By analogy with the classical case, we set $A_0 = A \cap \mathrm{Ker}(\Phi)$. For example, if $A = H^\infty(\mathbb{D})$ then $A_0 = \{f \in H^\infty(\mathbb{D}) : f(0) = 0\}$. For subdiagonal algebras the analogue of $H^p$ is $[A]_p$, the closure of $A$ in the noncommutative $L^p$ space $L^p(M)$, for $p \geq 1$. The latter object may be defined to be the completion of $M$ in the norm $\tau(|\cdot|^p)^{\frac{1}{p}}$. The spaces $L^p(M)$ are Banach spaces satisfying the usual duality relations

[1] This means that $\tau \circ \Phi = \tau$.

and Hölder inequalities [**15, 46**]. There is a useful alternative definition. For our (finite) von Neumann algebra $M$ on a Hilbert space $H$, define $\widetilde{M}$ to be the set of unbounded, but closed and densely defined, operators $T$ on $H$ which are affiliated to $M$ (that is, $Tu = uT$ for all unitaries $u \in M'$). This is a $*$-algebra with respect to the 'strong' sum and product (see Theorem 28 and the example following it in [**58**]). The trace $\tau$ extends naturally to the positive operators in $\widetilde{M}$. If $1 \leq p < \infty$, then $L^p(M, \tau) = \{a \in \widetilde{M} : \tau(|a|^p) < \infty\}$, equipped with the norm $\|\cdot\|_p = \tau(|\cdot|^p)^{1/p}$ (see e.g. [**45, 15, 58, 46**]). For brevity, we will in the following write $L^p$ or $L^p(M)$ for $L^p(M, \tau)$. Note that $L^1(M)$ is canonically isometrically isomorphic, via an $M$-module map, to the predual of $M$. Of course this isomorphism takes $T \in L^1(M)$ to the normal functional $\tau(T\,\cdot\,)$ on $M$. This isomorphism also respects the natural positive cones on these spaces (the natural cone of the predual of $M$ is the space of positive normal functionals on $M$).

Arveson realized that the appropriate Szegö theorem/formula for his algebras should read:

$$\Delta(h) = \inf\{\tau(h|a + d|^2) : a \in A_0, d \in \mathcal{D}, \Delta(d) \geq 1\}$$

for all $h \in L^1(M)_+$. Here $\Delta$ is the *Fuglede-Kadison determinant*, originally defined on $M$ by $\Delta(a) = \exp \tau(\log |a|)$ if $|a|$ is strictly positive, and otherwise, $\Delta(a) = \inf \Delta(|a| + \epsilon 1)$, the infimum taken over all scalars $\epsilon > 0$. Classically this quantity of course represents the geometric mean of elements of $L^\infty$. In [**7**] we noted that this definition of $\Delta(h)$ makes sense for $h \in L^1(M)$, and this form was used extensively in that paper. Although in [**2**], Arveson does define $\Delta$ for normal functionals (equivalently elements of $L^1(M)$), the above is not his original definition, and some work is necessary to prove that the two are equivalent (see [**7**, Section 2]).

In passing, we remark that the definition above of $\Delta(h)$ makes perfect sense for $h$ in any $L^q(M)$ where $q > 0$. Since we will need this later we quickly explain this point in our setting (see also [**12, 18**]), adapting the argument in the third paragraph of [**8**, Section 2]. We make use of the Borel functional calculus for unbounded operators applied to the inequality

$$0 \leq \log t \leq \frac{1}{q} t^q \quad t \in [1, \infty).$$

Notice that for any $0 < \epsilon < 1$, the function $\log t$ is bounded on $[\epsilon, 1]$. So given $h \in L^1(M)_+$ with $h \geq \epsilon$, it follows that $(\log h) e_{[0,1]}$ is similarly bounded. Moreover the previous centered equation ensures that $0 \leq (\log h) e_{[1,\infty)} \leq \frac{1}{q} h^q e_{[1,\infty)} \leq \frac{1}{q} h^q$. Here $e_{[0,\lambda]}$ denotes the spectral resolution of $h$. Thus if $h \in L^q(M)$ and $h \geq \epsilon$ then $\log h \in L^1(M)$.

Unfortunately, the conjectured noncommutative Szegö formula stated above, and the (no doubt more important) associated *Jensen's inequality*

$$\Delta(\Phi(a)) \leq \Delta(a), \qquad a \in A,$$

and *Jensen formula*

$$\Delta(\Phi(a)) = \Delta(a), \qquad \text{invertible } a \in A,$$

resisted proof for nearly 40 years (although Arveson did prove these for most of the examples that he was interested in). In 2004, via a judicious use of a noncommutative variant of a classical limit formula for the geometric mean, and a careful choice of recursively defined approximants, the second author proved in [**34**] that

all maximal subdiagonal algebras satisfy Jensen's formula (and hence the Szegö formula and Jensen's inequality too by Arveson's work). Settling this old open problem opened up the theory to the recent developments surveyed here. Of course much of the classical theory had already been generalized to subdiagonal algebras in Arveson's original and seminal paper [**2**], and in the intervening decades following it. We mention for example the work of Zsidó, Exel, McAsey, Muhly, Saito (and his school in Japan), Marsalli and West, Nakazi and Watatani, Pisier, Xu, Randrianantoanina, and others (see our reference list below, and references therein). This work, together with the results mentioned below, yields a complete noncommutative generalization of all of the classical theory surveyed in [**55**]. Since much of this work has been surveyed recently in [**46**, Section 8], we will not attempt to survey this literature here.

As a first set of results, which may be regarded in some sense as a 'mnemonic' for the subject, we obtain the same cycle of theorems as in the classical case:

THEOREM 1.2. *For a tracial subalgebra $A$ of $M$, the following eight conditions are equivalent:*

(i) *$A$ is maximal subdiagonal, that is: $\overline{A + A^*}^{w*} = M$.*
(ii) *a) $L^2$-density of $A + A^*$ in $L^2(M)$; and b) the unique normal state extension property, that is: if $g \in L^1(M)_+$, $\tau(fg) = \tau(f)$ for all $f \in A$, then $g = 1$.*
(iii) *$A$ has factorization, that is: an element $b \in M_+$ is invertible iff $b = a^*a$ for an invertible $a \in A$.*
(iv) *$A$ is logmodular, that is: if $b \in M_+$ is invertible then $b$ is a uniform limit of terms of the form $a^*a$ for invertible $a \in A$.*
(v) *$A$ satisfies the Szegö formula above.*
(vi) *Beurling-like invariant subspace condition (described in Section 4).*
(vii) *Beurling-Nevanlinna factorization property, that is: every $f \in L^2(M)$ such that $\Delta(f) > 0$ has an 'inner-outer factorization' $f = uh$, with $u$ unitary and $h \in H^2$ outer (that is, $1 \in [hA]_2$).*
(viii) *Gleason-Whitney property: there is at most one normal Hahn-Banach extension to $M$ of any normal functional on $A$.*

It will be noted that there is an extra condition in (ii) that does not appear in the classical case. It is interesting that this extra condition took some years to remove in the classical case (compare [**55**] and [**26**]). Although we have not succeeded yet in removing it altogether in our case, we have made partial progress in this direction in [**9**, Section 2].

We have also been able to prove many other generalizations of the classical generalized function theory in addition to those already mentioned; for example the F & M Riesz theorem, $L^p$ versions of the Szegö formula, the Verblunsky/Kolmogorov-Krein extension of the Szegö formula, inner-outer factorization, etc. These are featured in the Sections 5-8. For example, in Section 7 we will generalize important aspects of the classical theory of outer functions to subdiagonal algebras. In this regard we note that $h \in L^p(M)$ is *outer* if $[hA]_p = H^p$. That is, $h \in H^p$, and $1 \in [hA]_p$. This definition is in line with e.g. Helson's definition of outers in the matrix valued case he considers in [**20**] (we thank Q. Xu for this observation). In Section 7 we will restrict our attention to the special case of antisymmetric subdiagonal algebras, where the theory of outer functions works out particularly transparently and tidily; the general case will be treated in the forthcoming work

[**10**]. It is worthwhile pointing out however, that there are many interesting antisymmetric maximal subdiagonal algebras besides the weak* Dirichlet algebras—see [**2**]. Our main results here state 1) that $h \in H^p$ is outer iff $\Delta(h) = \Delta(\Phi(h)) > 0$ (one direction of this is not quite true in the general case discussed in [**10**]); and 2) if $f \in L^p(M)$ with $\Delta(f) > 0$, then $f = uh$ for a unitary $u \in M$ and an outer $h \in H^p$ (we prove elsewhere that this is true also in the general, i.e. non-antisymmetric, case). In particular, $|f| = |h|$ for this outer $h \in H^p$, which solves an approximately thirty year old problem (see e.g. the discussion in [**37**, p. 386], or [**46**, Chapter 8], particularly lines 8-12 on p. 1497 of the latter reference). For the commutative case of these results see [**41**].

We end this introduction by mentioning that there are many other, more recent, generalizations of $H^\infty$, based around multivariable analogues of the Sz-Nagy-Foiaş model theory for contractions. Many prominent researchers are currently intensively pursuing these topics, for example Popescu, Arias; Arveson again; Ball and Vinnikov; Davidson and Power and their brilliant collaborators; and Muhly and Solel. See e.g. [**48, 40**] and references therein. In essence, the unilateral shift is replaced by left creation operators on some variant of Fock space. These generalizations are very important at the present time, and are evolving in many directions (with links to wavelets, quantum physics, conservative linear systems, and so on). Although these theories also contain variants of parts of the theory of $H^\infty$ of classical domains, so that in a superficial reading the endeavours may appear to be similar, in fact they are quite far removed, and indeed have nothing in common from a practical angle. For example, those other theories have nothing to do with (finite) von Neumann algebra techniques, which are absolutely key for us. So, for example, if one compares Popescu's theorem of Szegö type from [**48**, Theorem 1.3] with the Szegö theorem for subdiagonal algebras discussed here, one sees that they are only related in a very formal sense. It is unlikely that the theory of subdiagonal algebras will merge to any great extent with these other theories, but certain developments in one theory might philosophically inspire the other.

## 2. Two $L^p$-space tools

In this survey we will only be able to prove a selection of our results, and even then some of the proofs will be sketchy. This is not the forum for a full blown account, and also some of the proofs are quite technical. Nonetheless, it seems worthwhile to explain to a general audience a couple of the tools, each of which is used several times, and which are quite helpful in adapting proofs of some classical results involving integrals, to the noncommutative case. The first tool is a useful reduction to the classical case. This may be viewed as a principle of local commutativity for semifinite von Neumann algebras which furnishes a link between classical and noncommutative $L^p$ spaces. Suppose that $h \in L^1(M)$ is selfadjoint. One may of course view $h$ as a normal functional on $M$, but we will want instead to view $h$ as an unbounded selfadjoint operator on the same Hilbert space on which $M$ acts, as we indicated above. As we shall show, $h$ may be regarded as a function in a classical $L^1$ space. Let $M_0$ be the von Neumann algebra generated by $h$ (see e.g. [**29**, p. 349]). This is a commutative subalgebra of $M$, and it is the intersection of all von Neumann algebras with which $h$ is affiliated. Let $\psi = \tau_{|M_0}$. Since $\psi$ is a faithful normal state on $M_0$, it is a simple consequence of the Riesz representation theorem applied to $\psi$, that $M_0 \cong L^\infty(\Omega, \mu_\tau)$ $*$-algebraically, for a

measure space $\Omega$ and a Radon probability measure $\mu_\tau$. Also, $L^1(M_0) \subset L^1(M)$, and $L^1(M_0) \cong L^1(\Omega, \mu_\tau)$. Via these identifications, $\tau$ restricts to the integral $\int_\Omega \cdot \, d\mu_\tau$ on $L^1(\Omega, \mu_\tau)$, and $h$ becomes a real valued function in $L^1(\Omega, \mu_\tau)$. In particular $\Delta(h)$ also survives the passage to commutativity since it is clear from the above that $\Delta(h) = \inf_{\epsilon > 0} \int_\Omega \log(|h| + \epsilon 1) \, d\mu_\tau$. Thus for many purposes, we are now back in the classical situation.

The second technique we will use is 'weighted noncommutative $L^p$ spaces' $L^p(M, h)$. Here $h \in L^1(M)_+$. We define $L^2(M, h)$ to be the completion of $M$ in the inner product

$$\langle a, b \rangle_h = \tau(h^{1/2} b^* a h^{1/2}), \qquad a, b \in M.$$

Note that $L^2(M, h)$ can be identified unitarily, and as $M$-modules, with the closure of $Mh^{1/2}$ in $L^2(M)$. Let $a \mapsto \Psi_a$ be the canonical inclusion of $A$ in $L^2(M, h)$. There is a canonical normal $*$-homomorphism representing $M$ as an algebra of bounded operators on $L^2(M, h)$. Indeed define

$$\pi(b)\Psi_a = \Psi_{ba}, \qquad a, b \in M,$$

and then extend this action to all of $L^2(M, h)$. This is very closely connected to the famous notion of the 'standard form' or 'standard representation' of a von Neumann algebra (see e.g. [**57**]).

More generally, we define $L^p(M, h)$ to be the completion in $L^p(M)$ of $Mh^{\frac{1}{p}}$. Note that if $e$ is the support projection of a positive $x \in L^p(M)$ (that is, the smallest projection in $M$ such that $ex = x$, or equivalently, the projection onto the closure of the range of $x$, with $x$ regarded as an unbounded operator), then it is well known (see e.g. [**28**, Lemma 2.2]) that $L^p(M)e$ equals the closure in $L^p(M)$ of $Mx$. Hence $L^p(M, h) = L^p(M)e$, where $e$ is the support projection of $h$. Now for any projection $e \in M$ it is an easy exercise to prove that the dual of $L^p(M)e$ is $eL^q(M)$ (see e.g. [**28**]). It follows that the dual of $L^p(M, h)$ is the variant of $L^q(M, h)$ where we consider the completion of $h^{\frac{1}{q}} M$.

This procedure corresponds in the classical case, to a Radon-Nikodym derivative, or to 'weighting' a given measure.

## 3. The equivalences (i)–(v) in Theorem 1.2

In this section we indicate a somewhat simplified route through the equivalences (i)–(v) in Theorem 1.2 above, which are originally from [**7**]. For those familiar with [**7**], we remark that the approach here 1) avoids the use of Lemmas 3.3 and 5.1, and Corollary 4.7, from that paper, 2) proves some implications more directly, and 3) is more self-contained, avoiding some of the reliance on results from other papers. We will still need to quote [**13**] in one place, and we will need a few facts about the Fuglede-Kadison determinant from [**2**].

Amongst the circle of equivalences in Theorem 1.2, it is trivial that (iii) $\Rightarrow$ (iv), and fairly obvious that (i) $\Rightarrow$ (ii). Indeed if (i) holds, and if $g \in L^1(M)_+$ with $\tau(fg) = \tau(f)$ for all $f \in A$, then $g - 1 \in A_\perp$. Since $g - 1$ is selfadjoint we deduce that $g - 1 \in (A + A^*)_\perp = (0)$. Similarly, if $g \in L^2(M)$ with $g \perp A + A^*$, then since $g \in L^1(M)$, we see that $g = 0$. So (ii) holds.

We describe briefly some of the main ideas in the proof from [**7**] that (iv) $\Rightarrow$ (v). This is a slight generalization of the solution from [**34**] of Arveson's long outstanding problem as to whether (i) implied (v) and the Jensen formula/inequality mentioned

in Section 1. Arveson had proved that (i) implied (iii) (another proof is sketched below, which is longer than Arveson's but can be used to yield some other facts too), and had also noted that if (i) held then (v) was equivalent to the Jensen inequality or the Jensen's formula. By means of some technical refinements to these arguments, [**7**, Proposition 3.5] proves that the validity of Jensen's formula, Jensen's inequality, and the Szegö formula are progressively stronger statements, with all three being equivalent if $A$ is logmodular. To see that logmodularity indeed does imply the Szegö formula, one therefore need only adapt the argument from [**34**] to show that (iv) implies Jensen's formula (cf. [**7**, Proposition 3.1]).

Next we prove that (v) implies the unique normal state extension property (that is, (ii)b). Suppose that we are given an $h \in L^1(M)_+$, such that $\tau(ha) = \tau(a)$ for all $a \in A$. Then $\tau(ha) = 0$ for all $a \in A_0$, and hence also for all $a \in A_0^*$, since $\overline{\tau(ha^*)} = \tau(ha)$. If $a \in A_0$, and $d \in \mathcal{D}$, then

$$\tau(h|a+d|^2) = \tau(h|a|^2 + hd^*a + ha^*d + h|d|^2) = \tau(|a|h|a| + |d|^2) \geq \tau(|d|^2).$$

Appealing to the Szegö formula in (v), we deduce that

$$\Delta(h) \;=\; \inf\{\tau(|d|^2) : d \in \mathcal{D}, \Delta(d) \geq 1\} \;\leq\; 1.$$

By [**2**, 4.3.1], we have

$$\tau(|d|^2) \geq \Delta(|d|^2) = \Delta(|d|)^2 = \Delta(d)^2.$$

It follows that

$$\Delta(h) \;=\; \inf\{\tau(|d|^2) : d \in \mathcal{D}, \Delta(d) \geq 1\} \;=\; 1.$$

By hypothesis, we also have $\tau(h) = \tau(1) = 1$. We now reduce to the classical case as we described in the previous section, so that the von Neumann algebra generated by $h$ is $*$-isomorphic to $L^\infty(\Omega, \mu)$ $*$-algebraically, for a measure space $\Omega$ and a Radon probability measure $\mu$, and $h$ becomes a real-valued function in $L^1(\Omega, \mu)$. We have $\int_\Omega h\, d\mu = 1 = \exp(\int_\Omega \log h\, d\mu)$. It is an elementary exercise in real analysis to show that this forces $h = 1$ a.e. This proves (ii)b.

We now sketch the proof that (iii) implies (i). This requires three technical background facts, which we now state. In [**7**, Proposition 3.2] it is shown that (iii) implies that for $h \in L^1(M)$, we have a 'Szegö formula':

$$\Delta(h) \;=\; \inf\{\tau(|ha|) : a \text{ invertible in } A, \Delta(a) \geq 1\}.$$

This in turn is an extension of facts about the Fuglede-Kadison determinant from [**2**]. Since the proof is rather long (using the Borel functional calculus for unbounded selfadjoint operators affiliated to a von Neumann algebra) we omit the details. Next, one shows that the last displayed formula, together with Jensen's formula (which we already know to be a consequence of (iii)), implies that if $h \in L^1(M)$ with $\tau(ha) = 0$ for every $a \in A$, then $\Delta(1-h) \geq 1$. See [**7**, Lemma 3.3] for the short calculation. The third fact that we shall need is that, just as in the classical case, if $h \in L^1(M)$ is selfadjoint, and if for some $\delta > 0$ we have

$$\Delta(1 - th) \geq 1\,, \qquad t \in (-\delta, \delta),$$

then $h = 0$. This is a noncommutative version of an extremely elegant and useful lemma which seems to have been proved for $L^2$ functions by Hoffman [**25**, Lemma 6.6], and then extended to $L^1$ functions by R. Arens in what apparently was a private communication to Hoffman. The noncommutative version is proved by reducing to the classical case as we described in the previous section, so that the von Neumann algebra generated by $h$ is $*$-isomorphic to $L^\infty(\Omega, \mu)$ $*$-algebraically,

for a measure space $\Omega$ and a Radon probability measure $\mu$, and $h$ becomes a function in $L^1(\Omega, \mu)$. We are now back in the classical situation, and one may invoke the classical result mentioned above. See [**7**, Section 2] for more details if needed.

We now explain how these facts give the implication (iii) $\Rightarrow$ (i). To show that $A + A^*$ is weak* dense in $M$, it suffices to show that if $h \in (A + A^*)_\perp$ then $h = 0$. Since $A + A^*$ is selfadjoint, it is easy to see that $h \in (A + A^*)_\perp$ if and only if $h + h^* \in (A + A^*)_\perp$ and $i(h - h^*) \in (A + A^*)_\perp$. We may therefore assume that $h$ is selfadjoint. By the facts in the last paragraph, $\Delta(1 - th) \geq 1$ for every $t \in \mathbb{R}$, and this implies that $h = 0$.

We now complete this circle of equivalences by using weighted $L^2$ space arguments (discussed in Section 2) to show that (ii) implies (iii), and that (v) implies the $L^2$-density of $A + A^*$ in $L^2(M)$ (that is, (ii)a).

We say that $A$ is *$\tau$-maximal* if $A = \{x \in M : \tau(xA_0) = 0\}$. The following is essentially due to Arveson (as pointed out to us by Xu, the proof in [**2**] of 'factorization' essentially only uses the hypotheses 1 or 2 below; see also the method of proof of [**5**, Theorem 3.1]). However we give a somewhat different proof since the method will be used again immediately after the theorem.

THEOREM 3.1. *Let $A$ be a tracial subalgebra of $M$. Consider the following statements:*

(1) *$A$ satisfies $L^2$-density and the unique normal state extension property;*
(2) *$A$ satisfies $\tau$-maximality and the unique normal state extension property;*
(3) *$A$ has factorization.*

*The following implications hold:* (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

PROOF. (Sketch) To see that (1) $\Rightarrow$ (2) we will need the space $A_\infty = [A]_2 \cap M$. It is an exercise to show that $A_\infty$ is also a tracial subalgebra of $M$, with respect to the same expectation $\Phi$ (see [**7**, Theorem 4.4]). However, an adaption of a beautiful von Neumann algebraic argument of Exel's from [**13**], shows that if $A$ satisfies hypothesis (1) then there is no such properly larger tracial algebra. See the proof of [**7**, Theorem 5.2]. Thus $A = A_\infty$.

By $L^2$-density, $L^2(M) = [A]_2 \oplus [A_0^*]_2$. Thus if $x \in M$ with $\tau(xA_0) = 0$, then $x \in (A_0^*)^\perp = [A]_2$, so that $x \in A_\infty = A$. Hence $A$ is $\tau$-maximal.

The following proof that (2) $\Rightarrow$ (3) rests on a slight modification of the proof of [**7**, Theorem 4.6(a)]. If $A$ is a $\tau$-maximal tracial subalgebra of $M$, and if $b \in A_\infty$, then there exists a sequence $(a_n) \subset A$ with $L^2$-limit $b$. If $c \in A_0$, then $a_n c \to bc$ in $L^2(M)$, and

$$\tau(bc) \;=\; \lim_n \tau(a_n c) \;=\; 0.$$

Thus $b \in A$. Therefore $A_\infty = A$. Now suppose that in addition $A$ satisfies the unique normal state extension property. If $b \in M_+$ is invertible, we consider the weighted noncommutative $L^2$ spaces' $L^2(M, b)$. Let $p$ be the orthogonal projection of 1 onto the subspace $[A_0]_2$, taken with respect to the weighted inner product. In [**7**, Theorem 4.6(a)] it is shown that $(1 - p)b(1 - p^*) \in L^1(\mathcal{D})_+$, and that also $(1 - p)b(1 - p^*) \geq \epsilon 1$, for some $\epsilon > 0$. Thus this element has a bounded inverse in $\mathcal{D}$. Set $e = ((1 - p)b(1 - p^*))^{-\frac{1}{2}} \in \mathcal{D}$, and let $a = e(1 - p) \in [A]_2$. It is routine to see that $a$ is bounded, so that $a \in M$. Hence $a \in M \cap [A]_2 = A_\infty = A$. Since $1 = aba^*$, and since $M$ is a finite von Neumann algebra, we also have $1 = ba^*a$, so that $b^{-1} = |a|^2$.

For any $a_0 \in A_0$ one sees that

$$\tau(a^{-1}a_0) = \tau(b(1-p^*)ea_0) = 0,$$

since $ea_0 \in [A_0]_2$, and $1-p \perp [A_0]_2$ in the weighted inner product. Thus

$$a^{-1} \in \{x \in M : \tau(xA_0) = 0\} = A,$$

using $\tau$-maximality. We deduce that $A$ has factorization. □

Finally, we say a few words about the tricky implication (v) implies (ii)a; full details are given in [**7**, Proof of Theorem 4.6 (b)]. We suppose that $k \in L^2(M)$ is such that $\tau(k(A+A^*)) = 0$. We need to show that $k = 0$. Since $A + A^*$ is a self-adjoint subspace of $M$, we may assume that $k = k^*$. Then $1-k \in L^1(M)$, so that by an equivalent form of (v), given $\epsilon > 0$ there exists an invertible element $b \in M_+$ with $\Delta(b) \geq 1$ and $\tau(|(1-k)b|) < \Delta(1-k) + \epsilon$. In this case it is a bit more complicated, but one can modify the weighted $L^2$-space argument in the second half of the proof of Theorem 3.1, with $b$ replaced by $b^{-2}$, to one find an element $a \in A_\infty$ with $b^2 = a^*a$. This element $a$ is used to prove that $\Delta(1-k) \geq 1$ (we omit the details). Replacing $k$ by $tk$, where $t \in [-1,1]$, we conclude that $\Delta(1-tk) \geq 1$ for such $t$. Thus by the Arens-Hoffman lemma (see the discussion surrounding the centered equation immediately preceding Theorem 3.1), we have that $k = 0$ as required. Hence $A + A^*$ is norm-dense in $L^2(M)$.

We end this section with a brief remark concerning algebras with the unique normal state extension property (that is, (ii)b in Theorem 1.2), in hope that they (perhaps in conjunction with results in [**9**, Section 2]) lead to a resolution of the question as to whether (ii)a is really necessary in the list of equivalences in Theorem 1.2. The hope is to prove that (ii)b $\Rightarrow$ (ii)a. Now by [**7**, Theorem 4.6(a)] we know that if $A$ satisfies (ii)b of Theorem 1.2, then $A_\infty$ has 'partial factorization'; that is if $b \in M_+$ is invertible, then we can write $b = |a| = |c^*|$ for some elements $a, c \in A$ which are invertible in $M$, with in addition $\Phi(a)\Phi(a^{-1}) = \Phi(c)\Phi(c^{-1}) = 1$. Then of course $1 = b^2b^{-2} = a^*acc^*$, and so $acc^*a^* = 1$. That is, $ac$ is unitary. If one could show that it is possible to select $a$ and $c$ in such a way that $\Phi(ac) = 1$, it would then follow that $1 = ac$. To see this simply compute $\tau(|1-ac|^2)$. Thus $A_\infty$ would then have the factorization property (iii), which would ensure the density of $A_\infty + A_\infty^*$ in $L^2(M)$. Clearly this implies that $A + A^*$ is dense too. In fact even if all we could show is that $a, c$ can be selected so that $\Phi(ac)$ is unitary, we could then replace $a$ by $\widetilde{a} = \Phi(ac)a$, and argue as before to see that $\widetilde{a}c = 1$, hence that $A_\infty$ has the factorization property, and hence $A + A^*$ is dense in $L^2(M)$.

## 4. Beurling's invariant subspace theorem

We discuss (right) $A$-invariant subspaces, that is closed subspaces $K \subset L^p(M)$ with $KA \subset K$. For the sake of brevity we will therefore suppress the word *right* in the following. These have been studied by many authors e.g. [**37, 38, 42, 46, 52, 61**], with an eye to Beurling-type invariant subspace theorems, usually in the case that $p = 2$. We mention just two facts from this literature: Saito showed in [**52**] that any $A$-invariant subspace of $L^p(M)$ is the closure of the bounded elements which it contains. In [**42**], Nakazi and Watatani decompose any $A$-invariant subspace of $L^2(M)$ into three orthogonal pieces which they called types I, II, and III. In the case that the center of $M$ contains the center of $\mathcal{D}$, they proved that every type I

invariant subspace of $L^2(M)$ is of the form $uH^2$ for a partial isometry $u$. This is a generalization of the classical Beurling invariant subspace theorem.

Our investigation into $A$-invariant subspaces of $L^p(M)$ revealed the fact that the appropriate Beurling theorem and 'Wold decomposition' for such spaces, are intimately connected with the *$L^p$-modules* developed recently by Junge and Sherman [**28**], and their natural 'direct sum', known as the '$L^p$-column sum'. We explain these terms: If $X$ is a closed subspace of $L^p(M)$, and if $\{X_i : i \in I\}$ is a collection of closed subspaces of $X$, which together densely span $X$, with the property that $X_i^* X_j = \{0\}$ if $i \neq j$, then we say that $X$ is the *internal column $L^p$-sum* $\oplus_i^{col} X_i$. There is also an extrinsic definition of this sum (see e.g. the discussion at the start of Section 4 of [**8**]). An $L^p(\mathcal{D})$-module is a right $\mathcal{D}$-module with an $L^{\frac{p}{2}}(\mathcal{D})$-valued inner product, satisfying axioms resembling those of a Hilbert space or Hilbert $C^*$-module [**28**, Definition 3.1].

The key point, and the nicest feature of $L^p(\mathcal{D})$-modules, is that they may all be written as a column $L^p$-sum of modules with cyclic vectors, each summand of a very simple form. It is this simple form that gives the desired Beurling theorem. We recall that a vector $\xi$ in a right $\mathcal{D}$-module $K$ is *cyclic* if $\xi\mathcal{D}$ is dense in $K$, and *separating* if $d \mapsto \xi d$ is one-to-one. Thinking of our noncommutative $H^p$ theory as a simultaneous generalization of noncommutative $L^p$-spaces, and of classical $H^p$ spaces, it is in retrospect not surprising that the Beurling invariant subspace classification theorem, in our setting, should be connected to Junge and Sherman's classification of $L^p$-modules.

For our purposes, we prefer to initially decompose $A$-invariant subspaces of $L^p(M)$ into two and not three summands, which we call type 1 (= type I) and type 2, and which are defined below. This is intimately connected to the famous *Wold decomposition*. For motivational purposes we discuss the latter briefly. For example, suppose that $u$ is an isometry on a Hilbert space $H$, and that $K$ is a subspace of $H$ such that $uK \subset K$. If $A$ is the unital operator algebra generated by $u$, and $A_0$ the nonunital operator algebra generated by $u$, then $K$ is $A$-invariant. (The reader may keep in mind the case where $u$ is multiplication by the monomial $z$ on the circle $\mathbb{T}$; in this case $A = H^\infty(\mathbb{D})$, and $A_0$ is the algebra of functions vanishing at 0.) The subspace $W = K \ominus uK = K \ominus [A_0 K]$ is *wandering* in the classical sense that $u^n W \perp u^m W$ for unequal nonnegative integers $n, m$. The Wold decomposition writes $K = K_1 \oplus K_2$, an orthogonal direct sum, where $K_1 = [AW]_2$, a condition which matches what is called 'type 1' below. Also, $uK_2 = K_2$, which is equivalent to $[A_0 K_2]_2 = K_2$, a criterion which matches what is called 'type 2' below. We have $u$ unitary on $K_2$, whereas the restriction of $u$ to subspaces of $K_1$ is never unitary (see e.g. [**44**, Lemma 1.5.1]). The match with the definitions in the next paragraph is exact in the case where $A = H^\infty(\mathbb{D})$.

If $K$ is an $A$-invariant subspace of $L^2(M)$ then the *right wandering subspace* of $K$ is defined to be $W = K \ominus [KA_0]_2$ (see [**42**]). As above, $K$ is *type 1* if the right wandering subspace generates $K$ (that is, $[WA]_2 = K$), and *type 2* if the right wandering subspace is trivial (that is, $W = (0)$). For the case $p \neq 2$ we similarly say that $K$ is type 2 if $[KA_0]_p = K$, but we define the right wandering subspace $W$ of $K$, and type 1 subspaces, a little differently (see [**8**]). We omit the details in the case $p \neq 2$, but to ease the reader's mind, we point out that there is a very explicit type-preserving lattice isomorphism between the closed (weak*-closed, if $p = \infty$) right $A$-invariant subspaces of $L^p(M)$ and those of $L^2(M)$ (see [**8**, Lemma 4.2 &

Theorem 4.5]). Thus the theory of $A$-invariant subspaces of $L^p(M)$ relies on first achieving a clear understanding of the $p = 2$ case.

THEOREM 4.1. [**8**] *If $A$ is a maximal subdiagonal subalgebra of $M$, if $1 \leq p \leq \infty$, and if $K$ is a closed (indeed weak* closed, if $p = \infty$) right $A$-invariant subspace of $L^p(M)$, then:*

(1) *$K$ may be written uniquely as an (internal) $L^p$-column sum $K_1 \oplus^{col} K_2$ of a type 1 and a type 2 invariant subspace of $L^p(M)$, respectively.*
(2) *If $K \neq (0)$ then $K$ is type 1 if and only if $K = \oplus_i^{col} u_i H^p$, for $u_i$ partial isometries in $M$ with mutually orthogonal ranges and $|u_i| \in \mathcal{D}$.*
(3) *The right wandering subspace $W$ of $K$ is an $L^p(\mathcal{D})$-module in the sense of Junge and Sherman (indeed $W^*W \subset L^{p/2}(\mathcal{D})$).*

*Conversely, if $A$ is a tracial subalgebra of $M$ such that every $A$-invariant subspace of $L^2(M)$ satisfies* (1) *and* (2) *(resp.* (1) *and* (3)*), then $A$ is maximal subdiagonal.*

We show now why this is a generalization of Beurling's theorem. Indeed the proof of (a) below proves a classical generalization of Beurling's theorem in just four lines. We recall that $h$ is outer in $H^p$ iff $[hA]_p = H^p$.

PROPOSITION 4.2. *Let $A$ be an antisymmetric subdiagonal subalgebra of $M$, and let $1 \leq p < \infty$. Then*

(a) *Every right invariant subspace $K$ of $L^2(M)$ such that $[KA_0]_p \neq K$, is of the form $u[A]_p$, for a unitary $u$ in $M$,*
(b) *Whenever $f \in L^p(M)$ with $f \notin [fA_0]_p$, then $f = uh$, for a unitary $u$ and an outer $h \in H^p$,*

PROOF. (a) The subspace $K$ here is not type 2, and it follows from Theorem 4.1 that $K_1 \neq (0)$, and that there is a nonzero partial isometry $u \in K_1$ with $|u| \in \mathcal{D} = \mathbb{C}\,1$. Hence $u^*u = 1$, and so $u$ is a unitary in $M$. Thus $K_1 = uH^p$. Since $K_1^*K_2 = (0)$, we have $K_2 = (0)$, and so $K = uH^p$.

(a) $\Rightarrow$ (b) By (a), $[fA]_p = uH^p$ for a unitary $u \in M$. Thus $f = uh$ for $h \in H^p$. Also, $h$ is outer, since $u[hA]_p = [fA]_p = uH^p$, so that $[hA]_p = H^p$. □

Admittedly, our formulation of Beurling's invariant subspace theorem above is more complicated if $A$ is not antisymmetric. We remark that it can be shown that an $A$-invariant subspace $K$ is of the form $uH^p$ for a unitary $u \in M$, if and only if the right wandering subspace of $K$ is a so-called 'standard' representation of $\mathcal{D}$, or equivalently it has a nonzero separating and cyclic vector, for the right action of $\mathcal{D}$ [**8**, Corollary 1.2]. Or, alternatively, appealing to Nakazi and Watatani's result mentioned above, it will follow that if the center of $\mathcal{D}$ is contained in the center of $\mathcal{M}$, then every type 1 $A$-invariant subspace of $L^p(M)$ is of the form $uH^p$ for a partial isometry $u \in M$.

Part (b) of Proposition 4.2 is called *Beurling-Nevanlinna factorization*. In fact one can actually give a precise formula for the $u$ and the $h$ (see [**10**] and Corollary 7.10 below). There are also generalized Beurling-Nevanlinna factorizations in the case that $A$ is not antisymmetric, which we shall discuss in Section 7.

## 5. The F. and M. Riesz and Gleason-Whitney theorems

The classical F. and M. Riesz theorem states that if $\mu \in C(\mathbb{T})^*$ is a complex measure on the circle whose Fourier coefficients vanish on the negative integers (that

is, $\mu$ annihilates the trigonometric polynomials $e^{in\theta}$, for $n \in \mathbb{N}$), then $\mu$ is absolutely continuous with respect to Lebesgue measure (see e.g. p. 47 of [**25**]). This is known to fail for weak* Dirichlet algebras; and hence it will fail for subdiagonal algebras too. However there is an equivalent version of the theorem which is true for weak* Dirichlet algebras [**25, 55**], and it holds too for a maximal subdiagonal algebra $A$ in $M$, with $\mathcal{D}$ finite dimensional. Moreover, one can show that this dimension condition is necessary and sufficient for the theorem to hold (see [**9**]).

THEOREM 5.1. (Noncommutative F. and M. Riesz theorem) *If $A$ is as above, and if a functional $\varphi \in M^*$ annihilates $A_0$ then the normal part of $\varphi$ annihilates $A_0$, and the singular part annihilates $A$.*

Exel proved a striking 'norm topology' variant in 1990 [**14**], but unfortunately it does not apply to subdiagonal algebras, which involve some extra complications. The strategy of the proof is to translate the main ideas of the classical proof to the noncommutative context by means of a careful and lengthy Hilbert space analysis of the 'GNS' construction for a subdiagonal algebra, using the weighted $L^2$ spaces which we discussed in Section 2, noncommutative Lebesgue-Radon-Nikodym decomposition of states, etc.

The F. and M. Riesz theorem, like its classical counterpart, has many applications. The main one that we discuss here is the Gleason-Whitney theorem, and the equivalence with (viii) in Theorem 1.2.

If $A \subset M$ then we say that $A$ has property (GW1) if every extension to $M$ of any normal functional on $A$, keeping the same norm, is normal on $M$. We say that $A$ has property (GW2) if there is at most one normal extension to $M$ of any normal functional on $A$, keeping the same norm. We say that $A$ has the *Gleason-Whitney property* (GW) if it possesses (GW1) and (GW2). This is simply saying that there is a unique extension to $M$ keeping the same norm of any normal functional on $A$, and this extension is normal.

THEOREM 5.2. *If $A$ is a tracial subalgebra of $M$ then $A$ is maximal subdiagonal if and only if it possesses property (GW2). If $\mathcal{D}$ is finite dimensional, then $A$ is maximal subdiagonal if and only if it possesses property (GW).*

PROOF. We prove that if $\dim(\mathcal{D}) < \infty$, and if $A$ is maximal subdiagonal, then $A$ satisfies (GW1). See [**9**, Theorem 4.1] for proofs of the other statements.

Let $\rho$ be a norm-preserving extension of a normal functional $\omega$ on $A$. By basic functional analysis, $\omega$ is the restriction of a normal functional $\tilde{\omega}$ on $M$. We may write $\rho = \rho_n + \rho_s$, where $\rho_n$ and $\rho_s$ are respectively the normal and singular parts, and $\|\rho\| = \|\rho_n\| + \|\rho_s\|$. Then $\rho - \tilde{\omega}$ annihilates $A$, and hence by our F. and M. Riesz theorem both the normal and singular parts, $\rho_n - \tilde{\omega}$ and $\rho_s$ respectively, annihilate $A_0$. Hence they annihilate $A$, and in particular $\rho_n = \omega$ on $A$. But this implies that

$$\|\rho_n\| + \|\rho_s\| = \|\rho\| = \|\omega\| \leq \|\rho_n\|.$$

We conclude that $\rho_s = 0$. Thus $A$ satisfies (GW1). □

There is another (simpler) variant of the Gleason-Whitney theorem [**25**, p. 305], which is quite easy to prove from the F. & M. Riesz theorem:

THEOREM 5.3. *Let $A$ be a maximal subdiagonal subalgebra of $M$ with $\mathcal{D}$ finite dimensional. If $\omega$ is a normal functional on $M$ then $\omega$ is the unique Hahn-Banach extension of its restriction to $A + A^*$. In particular, $\|\omega\| = \|\omega_{|A+A^*}\|$.*

Corollary 5.4. (Kaplansky density theorem for subdiagonal algebras) *Let $A$ be a maximal subdiagonal subalgebra of $M$ with $\mathcal{D}$ finite dimensional. Then the unit ball of $A + A^*$ is weak* dense in* $\mathrm{Ball}(M)$.

Proof. If $C$ is the unit ball of $A + A^*$, it follows from the last theorem that the pre-polar of $C$ is $\mathrm{Ball}(M_\star)$. By the bipolar theorem, $C$ is weak* dense in $\mathrm{Ball}(M)$. □

## 6. The Fuglede-Kadison determinant and Szegö's theorem for $L^p(M)$

The Fuglede-Kadison determinant $\Delta$, and its amazing properties, is perhaps the main tool in the noncommutative $H^p$ theory. In fact the development of the theory demands a knowledge of the properties of this determinant for possibly unbounded operators (as one can see, for example, by the role played by Theorem 2.1 of [**7**] in our theory). In [**15**], Fuglede and Kadison study the determinant as a function on $M$. In Section 1 we defined $\Delta$ for elements of $L^q(M)$ for any $q > 0$. In fact, as was pointed out to us by Quanhua Xu, L. G. Brown investigated the determinant and its properties on more general domains in the early 1980s [**12**], and full proofs appear in the preprint of Haagerup and Schultz [**18**], for a much more general class of $\tau$-measurable operators. The properties which we shall need are collected in the following result, which we shall use, often silently, in the next few sections. For the readers convenience we show how these properties may be deduced from [**7**, Theorem 2.1], which in turn is an $L^1$ version of a related determinant formula of Arveson.

Theorem 6.1. *If $p > 0$ and $h \in L^p(M)$ then*

(1) $\Delta(h) = \Delta(h^*) = \Delta(|h|)$.
(2) *If $h \geq g$ in $L^p(M)_+$ then $\Delta(h) \geq \Delta(g)$.*
(3) *If $h \geq 0$ then $\Delta(h^q) = \Delta(h)^q$ for any $q > 0$.*
(4) $\Delta(hb) = \Delta(h)\Delta(b) = \Delta(bh)$ *for any $b \in L^q(M)$ and any $q > 0$.*

Proof. (1) By [**15**, 2.5(ii) and (iv), 2.8] we have $\tau(\log(|h|+1)) = \tau(\log(|h^*|+1)$. Since $\log(t+\epsilon) = \log(\epsilon^{-1}t + 1) + \log(\epsilon)$, we conclude that $\tau(\log(|h| + \epsilon 1)) = \tau(\log(|h^*| + \epsilon 1)$ for any $\epsilon > 0$. Then (1) follows from the definition of $\Delta(h)$.

(2) Observe that [**15**, 2.5(iii) and (iv), 2.8] ensures that $\tau(\log(\delta h + 1)) \geq \tau(\log(\delta g + 1))$ for any $\delta > 0$. Arguing as in (1) above, we may conclude from this that $\tau(\log(h + \epsilon 1)) \geq \tau(\log(g + \epsilon 1))$ for any $\epsilon > 0$, and hence that $\Delta(h) \geq \Delta(g)$.

(3) First suppose that $h \geq \epsilon 1$ for some $\epsilon > 0$. Then by the Borel functional calculus for affiliated positive operators, $\Delta(h^q) = \exp \tau(\log(h^q)) = \exp \tau(q \log(h)) = (\exp \tau(\log(h)))^q = \Delta(h)^q$ as required. For general $h \in L^p(M)_+$ it follows from this fact and (2) that $\Delta(h^q + \epsilon 1) = \Delta((h^q + \epsilon 1)^{1/q})^q \geq \Delta(h)^q$ for any $\epsilon > 0$. Thus on taking the infimum, it is clear that $\Delta(h^q) \geq \Delta(h)^q$. Since both $q > 0$ and $p > 0$ were arbitrary, we also have that $\Delta(h) \geq \Delta(h^q)^{1/q}$. The claim follows.

(4) It suffices to prove the second equality, and that only in the case that $h$ and $b$ are positive. Indeed if that case held, then using (1) many times we have

$$\Delta(bh) = \Delta(|bh|) = \Delta(|b|h) = \Delta(h^*|b|) = \Delta(|h^*||b|) = \Delta(|h^*|)\Delta(b) = \Delta(b)\Delta(h),$$

for general $h, b$, giving in turn that:

$$\Delta(hb) = \Delta(b^*h^*) = \Delta(b^*)\Delta(h^*) = \Delta(b)\Delta(h).$$

Assume first that $b \in M$ and that $b$ is invertible, with $h \geq \epsilon 1$ for some $\epsilon > 0$. Then $h^{1/p} \geq \epsilon^{1/p} 1$. Let $h_n = h e_{[0,n)} \in M$ and $\widetilde{h_n} = h e_{[0,n)} + \epsilon e_{[n,\infty)} \in M$. From the first part of the proof of [**7**, Theorem 2.1] applied to $h^{1/p}$, one sees that

$$\tau(\log(\widetilde{h_n}^{1/p})) = \tau(\log(h_n^{1/p} + \epsilon^{1/p} e_{[n,\infty)})) = \tau(\log(h^{1/p}) e_{[0,n)}) + \log \epsilon^{1/p} \tau(e_{[n,\infty)}),$$

and this converges to $\tau(\log(h^{1/p}))$. Hence $\Delta(\widetilde{h_n}^{1/p}) \to \Delta(h^{1/p})$ as $n \to \infty$. We are using the fact that $\chi_{[0,n)}(h^{1/p}) = \chi_{[0,n)}(h)^{1/p}$, by the functional calculus. Thus $\Delta(\widetilde{h_n}) \to \Delta(h)$ by (3) above. By construction $|hb| \geq |\widetilde{h_n} b|$, and consequently,

$$\Delta(hb) = \Delta(|hb|) \geq \Delta(|\widetilde{h_n} b|) = \Delta(\widetilde{h_n} b) = \Delta(\widetilde{h_n}) \Delta(b).$$

Letting $n \to \infty$ yields $\Delta(hb) \geq \Delta(h)\Delta(b)$.

For any $h \geq 0$, invertible $b \in M$, and $\epsilon > 0$, the last paragraph shows that

$$\Delta((h + \epsilon 1) b) \geq \Delta(h + \epsilon 1)\Delta(b) \geq \Delta(h)\Delta(b).$$

If we take the infimum over $\epsilon > 0$, we get $\Delta(hb) \geq \Delta(h)\Delta(b)$ as before. To see this, let $E$ be the set of invertible $x \in M_+$ with $\Delta(x) \geq 1$. If $p \geq 1$ we can embed $h$ into $L^1(M)$ and use equation (2.2) in [**7**] twice, to see that $\inf_{\epsilon > 0} \Delta((h + \epsilon 1) b)$ equals

$$\inf_{\epsilon > 0} \inf_{x \in E} \tau(|(h + \epsilon 1) b x|) = \inf_{x \in E} \inf_{\epsilon > 0} \tau(|(h + \epsilon 1) b x|) = \inf_{x \in E} \tau(|hbx|) = \Delta(hb).$$

If $1 > p > 0$, we firstly notice that $(h + \epsilon 1) b$ tends to $hb$ in $L^p(M)$, as $\epsilon$ decreases to 0. But then $\|(h + \epsilon 1) b\|_p^p$ tends to $\|hb\|_p^p$ as $\epsilon$ decreases to 0 (see Theorems 3.7 or 4.9(iii) in [**15**]). Since for any $1 > p > 0$, the fact that $|(h + \epsilon 1) b| \geq |hb|$ forces $|(h + \epsilon 1) b|^p \geq |hb|^p$ (see e.g. [**54**, Lemma 2.3]), convergence of the $p$-quasinorms translates to the fact that $|(h + \epsilon 1) b|^p$ decreases to $|hb|^p$ in $L^1(M)$-norm as $\epsilon$ decreases to 0. Arguing as before it now follows from equation (2.2) in [**7**] that $\inf_{\epsilon > 0} \Delta(|(h + \epsilon 1) b|^p)$ equals

$$\inf_{\epsilon > 0} \inf_{x \in E} \tau(||(h + \epsilon 1) b|^p x|) = \inf_{x \in E} \inf_{\epsilon > 0} \tau(||(h + \epsilon 1) b|^p x|) = \inf_{x \in E} \tau(||hb|^p x|) = \Delta(|hb|^p).$$

By (1) and (3) this is the same as saying that $\inf_{\epsilon > 0} \Delta((h + \epsilon 1) b) = \Delta(hb)$.

Having established that $\Delta(hb) \geq \Delta(h)\Delta(b)$ for any $h \in L^p(M)$ and any invertible $b \in M_+$, we can get the converse inequality by using the above to show

$$\Delta(h) = \Delta((hb)b^{-1}) \geq \Delta(hb)\Delta(b^{-1}) = \Delta(hb)\Delta(b)^{-1}.$$

Thus $\Delta(hb) = \Delta(h)\Delta(b)$.

For general $b \in M_+$, we have from the above that $\Delta(h(b + \epsilon 1)) = \Delta(h)\Delta(b + \epsilon 1)$ for any $\epsilon > 0$. Taking the infimum over $\epsilon$, the argument used above easily adapts to show that $\inf_{\epsilon > 0} \Delta((b + \epsilon 1) h) = \Delta(bh)$, and hence (on taking adjoints) that $\inf_{\epsilon > 0} \Delta(h(b + \epsilon 1)) = \Delta(hb)$. Thus $\Delta(hb) = \Delta(h)\Delta(b)$ as required.

Now let $b \in L^q(M)_+$ with $0 < q < \infty$. As before first assume that $h \geq \epsilon 1$ for some $\epsilon > 0$. Then there exists $k \in M_+$ such that $1 = kh$. By what we proved for the case $q = \infty$ we have $1 = \Delta(kh) = \Delta(k)\Delta(h)$, and hence $\Delta(k)^{-1} = \Delta(h)$. Thus on again using this result,

$$\Delta(h)\Delta(b) = \Delta(h)\Delta(khb) = \Delta(h)\Delta(k)\Delta(hb) = \Delta(hb),$$

as required. For general $h \in L^p(M)_+$, it follows from what we have just proved, that $\Delta((h + \epsilon 1) b) = \Delta(h + \epsilon 1)\Delta(b)$. We may now once again easily adapt the earlier argument to see that $\Delta(hb) = \inf_{\epsilon > 0} \Delta((h + \epsilon 1) b)$. In the present circumstance, given $r > 0$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, it is the cases $r \geq 1$ and $1 > r > 0$ that need to be dealt with separately rather than $p \geq 1$ and $1 > p > 0$. Specifically, $(h + \epsilon 1) b \to hb$

with respect to $\|\cdot\|_r$ as $\epsilon \to 0$. If $r \geq 1$, we have $(h+\epsilon 1)b, hb \in L^1(M)$, in which case the same argument as before shows that

$$\inf_{\epsilon>0} \Delta((h+\epsilon 1)b) = \inf_{\epsilon>0} \inf_{x\in E} \tau(|(h+\epsilon 1)bx|) = \Delta(hb).$$

If, on the other hand, $1 > r > 0$, then from the previously mentioned convergence with respect to $\|\cdot\|_r$, it follows that $\|(h+\epsilon 1)b\|_r^r$ tends to $\|hb\|_r^r$ as $\epsilon \to 0$ (see Theorems 3.7 or 4.9(iii) in [**15**]). As before, $|(h+\epsilon 1)b| \geq |hb|$ forces $|(h+\epsilon 1)b|^r \geq |hb|^r$, and so $\tau(|(h+\epsilon 1)b|^r - |hb|^r) = \tau(||(h+\epsilon 1)b|^r - |hb|^r|) \to 0$. Thus $|(h+\epsilon 1)b|^r \to |hb|^r$ in $L^1(M)$-norm as $\epsilon \to 0$. Consequently, $|(h+\epsilon 1)b|^r x \to |hb|^r x$ in $L^1(M)$-norm for any $x \in M$. Repeating the idea in the sixth centered equation of the present proof, shows that $\inf_{\epsilon>0} \Delta(|(h+\epsilon 1)b|^r) = \Delta(|hb|^r)$. Thus by (1) and (3) $\Delta(hb) = \inf_{\epsilon>0} \Delta((h+\epsilon 1)b)$ as required, and we finish as before. □

**Remark.** We take the opportunity here to use a fact from the last proof to correct what appears to be a small gap early in the proof of [**7**, Theorem 2.1]. This occurs around at the point where it is proved that $\Delta(h)$ is $\leq$ the infimum in (2.1) there. One can fill the gap by replacing $he_{[0,n)}$ in parts of that calculation by what is called $\widetilde{h_n}$ above; and use the fact, proved above, that $\Delta(\widetilde{h_n}) \to \Delta(h)$.

**Throughout the rest of this section, $A$ is a maximal subdiagonal algebra in $M$.**

It is proved in [**9**] that for $h \in L^1(M)_+$ and $1 \leq p < \infty$, we have

$$\Delta(h) = \inf\{\tau(h|a+d|^p) : a \in A_0, d \in \mathcal{D}, \Delta(d) \geq 1\}.$$

A perhaps more useful variant of this formula is as follows:

THEOREM 6.2. [**10**] *If $h \in L^q(M)_+$ and $0 < p, q < \infty$, we have*

$$\begin{aligned} \Delta(h) &= \inf\{\tau(|h^{\frac{q}{p}} a|^p)^{\frac{1}{q}} : a \in A, \Delta(\Phi(a)) \geq 1\} \\ &= \inf\{\tau(|a h^{\frac{q}{p}}|^p)^{\frac{1}{q}} : a \in A, \Delta(\Phi(a)) \geq 1\}. \end{aligned}$$

*The infimums are unchanged if we also require $a$ to be invertible in $A$, or if we require $\Phi(a)$ to be invertible in $\mathcal{D}$.*

COROLLARY 6.3. (Generalized Jensen inequality) [**10**] *Let $A$ be a maximal subdiagonal algebra. For any $h \in H^1$ we have $\Delta(h) \geq \Delta(\Phi(h))$.*

We recall that although $L^p(M)$ is not a normed space if $1 > p > 0$, it is a so-called linear metric space with metric given by $\|x-y\|_p^p$ for any $x, y \in L^p$ (see [**15**, 4.9]). Thus although the unit ball may not be convex, continuity still respects all elementary linear operations.

COROLLARY 6.4. *Let $h \in L^q(M)_+$ and $0 < p, q < \infty$. If $h^{\frac{q}{p}} \in [h^{\frac{q}{p}} A_0]_p$, then $\Delta(h) = 0$. Conversely, if $A$ is antisymmetric and $\Delta(h) = 0$, then $h^{\frac{q}{p}} \in [h^{\frac{q}{p}} A_0]_p$. Indeed if $A$ is antisymmetric, then*

$$\Delta(h) = \inf\{\tau(|h^{\frac{q}{p}}(1-a_0)|^p)^{\frac{1}{q}} : a_0 \in A_0\}.$$

PROOF. The first assertion follows by taking $a$ in the infimum in Theorem 6.2 to be of the form $1 - a_0$ for $a_0 \in A_0$.

If $A$ is antisymmetric, and if $t \geq 1$ with $\tau(|h^{\frac{q}{p}}(t1 + a_0)|^p)^{\frac{1}{q}} < \Delta(h) + \epsilon$, then $\tau(|h^{\frac{q}{p}}(1 + a_0/t)|^p)^{\frac{1}{q}} < \Delta(h) + \epsilon$. From this the last assertion follows that the infimums in Theorem 6.2 can be taken over terms of the form $1 + a_0$ where $a_0 \in A_0$.

If this infimum was 0 we could then find a sequence $a_n \in A_0$ with $h^{\frac{q}{p}}(1+a_n) \to 0$ with respect to $\|\cdot\|_p$. Thus $h^{\frac{q}{p}} \in [h^{\frac{q}{p}} A_0]_p$. $\square$

**Remark.** The converse in the last result is false for general maximal subdiagonal algebras (e.g. consider $A = M = M_n$).

The classical strengthening of Szegö's formula, to the case of general positive linear functionals, extends even to the noncommutative context. Although the classical version of this theorem is usually attributed to Kolmogorov and Krein, we have been informed by Barry Simon that Verblunsky proved it in the mid 1930's (see e.g. [**59**]):

THEOREM 6.5. [**10**] (Noncommutative Szegö-Verblunsky-Kolmogorov-Krein theorem) *Let $\omega$ be a positive linear functional on $M$, and let $\omega_n$ and $\omega_s$ be its normal and singular parts respectively, with $\omega_n = \tau(h\,\cdot)$ for $h \in L^1(M)_+$. If $\dim(\mathcal{D}) < \infty$, then*

$$\Delta(h) = \inf\{\omega(|a|^2) : a \in A, \Delta(\Phi(a)) \geq 1\}.$$

After seeing this result, Xu was able to use our Szegö formula, and facts about singular states, to remove the hypothesis that $\dim(\mathcal{D}) < \infty$, and to replace the '2' by a general $p$. We briefly sketch Xu's proof for the case $p = 2$ (see [**5**] for full details). Firstly note that the fact that $A$ has factorization forces the sets $\{|a|^2 : a \in A, \Delta(\Phi(a)) \geq 1\}$ and $\{x : x \in M^{-1} \cap M_+, \Delta(x)) \geq 1\}$ to have a common closure. Hence for any continuous linear functional $\rho$ on $M$, we obtain

$$\inf\{\rho(|a|^2) : a \in A, \Delta(\Phi(a)) \geq 1\} = \inf\{\rho(x) : x \in M^{-1} \cap M^+, \Delta(x)) \geq 1\}.$$

(See for example [**2**, 4.4.3].) It is clear that

$$\inf\{\omega_n(|a|^2) : a \in A, \Delta(\Phi(a)) \geq 1\} \leq \inf\{\omega(|a|^2) : a \in A, \Delta(\Phi(a)) \geq 1\}.$$

Hence if we can show that $\inf\ \omega_n(x) \geq \inf\ \omega(x)$, these infimums taken over the set of $x \in M^{-1} \cap M^+, \Delta(x)) \geq 1$, then the previously centered equality combined with the Szegö formula, will ensure that $\Delta(h) = \inf\{\omega(|a|^2) : a \in A, \Delta(\Phi(a)) \geq 1\}$ as required. By [**56**, Theorem III.3.8] there exists an increasing net $(e_i)$ of projections in the kernel of $\omega_s$ such that $e_i \to 1$ strongly. Given $\epsilon > 0$, set

$$x_i = \epsilon^{\tau(e_i)-1}(e_i + \epsilon e_i^{\perp}).$$

Clearly $x_i \in M^{-1} \cap M^+$. Moreover by direct computation $\Delta(x_i) = 1$. Now let $x \in M^{-1} \cap M^+$ be given with $\Delta(x) \geq 1$. The strong convergence of the $e_i$'s to 1 ensures that $x_i x x_i$ converges to $1x1 = x$ in the weak* topology. The weak* continuity of $\omega_n$ then further ensures that

$$\limsup \omega(x_i x x_i) = \omega_n(x) + \limsup \omega(x_i x x_i) \leq \omega_n(x) + \epsilon^2 \|x\| \omega_s(1).$$

(Here we have made use of the facts that $\omega_s(e_i) = 0$, and that $x_i x x_i$ is dominated by $\|x\| \epsilon^{2(\tau(e_i)-1)}(e_i + \epsilon^2 e_i^{\perp})$.) Since $\epsilon > 0$ is arbitrary, and since $\limsup \omega(x_i x x_i)$ dominates the infimum of $\omega(x)$, for $x \in M^{-1} \cap M^+, \Delta(x)) \geq 1\}$, the required inequality follows.

## 7. Inner-outer factorization and the characterization of outers

This section is entirely composed of very recent results, and we include almost all the proofs. In most of the section $A$ is an antisymmetric maximal subdiagonal algebra; the much more complicated general case is discussed in more detail in [**10**]. We recall that if $h \in H^1$ then $h$ is *outer* if $[hA]_1 = H^1$. An *inner* is a unitary which happens to be in $A$. We assume $p \geq 1$ throughout this section.

LEMMA 7.1. *Let $1 \leq p \leq \infty$, and let $A$ be a maximal subdiagonal algebra. Then $h \in L^p(M)$ and $h$ is outer in $H^1$, iff $[hA]_p = H^p$. (Note that $[\cdot]_\infty$ is the weak* closure.)*

*If these hold, then $h \notin [hA_0]_p$.*

PROOF. It is obvious that if $[hA]_p = H^p$ then $[hA]_1 = H^1$. Conversely, if $[hA]_1 = H^1$ and $h \in L^p(M)$, then by [**8**, Lemma 4.2] (the proof of the assertion we are using works for all $p$) and [**52**, Proposition 2], we have

$$[hA]_p = [hA]_1 \cap L^p(M) = H^1 \cap L^p(M) = H^p.$$

If $h \in [hA_0]_p$ then $1 \in [hA]_p \subset [[hA_0]_p A]_p \subset [hA_0]_p$. Now $\Phi$ continuously extends to a map which contractively maps $L^p(M)$ onto $L^p(\mathcal{D})$ (see e.g. Proposition 3.9 of [**38**]). If $ha_n \to 1$ in $L^p$, with $a_n \in A_0$, then

$$0 = \Phi(h)\Phi(a_n) = \Phi(ha_n) \to \Phi(1) = 1,$$

This forces $\Phi(1) = 0$, a contradiction. □

LEMMA 7.2. *Let $A$ be a maximal subdiagonal algebra. If $h \in H^1$ is outer then as an unbounded operator $h$ has dense range and trivial kernel. Thus $h = u|h|$ for a unitary $u \in M$. Also, $\Phi(h)$ has dense range and trivial kernel.*

PROOF. If $h$ is considered as an unbounded operator, and if $p$ is the range projection of $h$, then since there exists a sequence $(a_n)$ in $A$ with $ha_n \to 1$ in $L^1$-norm, we have that $p^\perp = 0$. Thus the partial isometry $u$ in the polar decomposition of $h$ is surjective, and hence $u$ is a unitary in $M$. It follows that $|h|$ has dense range, and hence it, and $h$ also, have trivial kernel.

For the last part note that

$$L^1(\mathcal{D}) = \Phi(H^1) = \Phi([hA]_1) = [\Phi(h)\mathcal{D}]_1.$$

Thus we can apply the above arguments to $\Phi(h)$ too. □

There is a natural equivalence relation on outers. The following is proved similarly to the classical case:

PROPOSITION 7.3. *If $h \in H^p$ is outer, and if $u$ is a unitary in $\mathcal{D}$, then $h' = uh$ is outer in $H^p$ too. If $h, k \in H^p$ are outer, then $|h| = |k|$ iff there is unitary $u \in \mathcal{D}$ with $h = uk$. Such $u$ is unique.*

As in the classical case, if $h \in H^2$ is outer, then $h^2$ is outer in $H^1$. Indeed one may follow the proof in [**55**, p. 229], and the same proof shows that a product of any two outers is outer. We do not know at the time of writing whether every outer in $H^1$ is the square of an outer in $H^2$.

We now move towards a generalization of a beautiful classical characterization of outer functions. We will need Marsalli and West's important earlier factorization result generalizing the classical Riesz factorization. This states that given any

$1 \leq p, q, r \leq \infty$ with $1/p + 1/q = 1/r$, and $x \in H^r$, then $x = yz$ for some $y \in H^p$ and $z \in H^q$ [**38**]. (Marsalli and West actually only consider the case $r = 1$, but their proof generalises readily. A slightly sharpened version of their result may be found in [**32**].)

THEOREM 7.4. *If $A$ is an antisymmetric subdiagonal algebra then $h \in H^p$ is outer iff $\Delta(h) = |\tau(h)| > 0$.*

PROOF. The case that $p = 2$ follows exactly as in the proof of [**25**, Theorem 5.6]. The case for general $p$ follows from the $p = 1$ case and Lemma 7.1. Hence we may suppose that $p = 1$. By the Riesz factorization mentioned above, $h = h_1 h_2$ for $h_1, h_2 \in H^2$. Since $\Delta(h_1)\Delta(h_2) = \Delta(h) = \Delta(\Phi(h)) = \Delta(\Phi(h_1))\Delta(\Phi(h_2)) > 0$ and $\Delta(h_i) \geq \Delta(\Phi(h_i))$ for each $i = 1, 2$ (by the generalized Jensen inequality), we must have $\Delta(h_i) = \Delta(\Phi(h_i)) > 0$ for each $i = 1, 2$. Thus both $h_1$ and $h_2$ must be outer elements of $[A]_2$, by the first line of the proof. However we said above that a product of outers is outer.

Finally, if $h$ is outer then $1 = \lim_n ha_n$ where $a_n \in A$, so that $1 = \lim_n \tau(h)\tau(a_n)$, forcing $\tau(h) \neq 0$. By the generalized Jensen inequality, $\Delta(h) \geq |\tau(h)|$. By Corollary 6.4, for any $a_0 \in A_0$ we have

$$\Delta(h) \leq \tau(||h|(1 - a_0)|) = \tau(|h - ha_0|).$$

Since $h$ is outer, it is easy to see that $[hA_0]_1 = [A_0]_1$, and thus we may replace $ha_0$ in the last inequality by $h - \tau(h)1 \in [A_0]_1$. Thus $\Delta(h) \leq \tau(|\tau(h)1|) = |\tau(h)|$. □

**Remarks.** 1) In the general non-antisymmetric case, one direction of the last assertion of the last theorem is not quite true as written. Indeed in the case that $A = M = L^\infty[0, 1]$, then outer functions in $L^2$ are exactly the ones which are a.e. nonzero. One can easily find an increasing function $h : [0, 1] \to (0, 1]$, which satisfies $\Delta(h) = 0$ or equivalently $\int_0^1 \log h = -\infty$. In the example of the upper triangular matrices $\mathcal{T}_2$ in $M_2$, it is easy to find outers $h$ (i.e. invertibles in $\mathcal{T}_2$) with $0 < |\tau(h)| < \Delta(h)$.

2) As in [**55**, 2.5.1 and 2.5.2], $h$ is outer in $H^2$ iff $|h| - \tau(h)u^* \in [|h|A_0]_2$, where $u$ is the partial isometry in the polar decomposition of $h$, and iff $|h| - \tau(h)u^*$ is the projection of $|h|$ onto $[|h|A_0]_2$. There may be a gap in the proof of [**55**, 2.5.3]. The following principle which they seem to be using is not correct: If $K$ is a subspace of a Hilbert space $H$, and if $\xi \notin K$, and $\xi - \xi_0 \in H \ominus K$ with $\|\xi - \xi_0\| = d(\xi, K)$, then $\xi_0$ is the projection of $\xi$ onto $K$.

As a byproduct of the method in the proof of the last theorem one also has:

COROLLARY 7.5. *Suppose that $A$ is antisymmetric. An element in $H^r$ is outer iff it is the product of an outer in $H^p$ and an outer in $H^q$, whenever $1 \leq p, q, r \leq \infty$ with $1/p + 1/q = 1/r$.*

COROLLARY 7.6. *If $A$ is antisymmetric, then $h$ is outer in $H^p$ iff $[Ah]_p = H^p$.*

PROOF. Replacing $A$ by $A^*$, we see that $\Delta(h) = \Delta(h^*) > 0$ is equivalent to $h^*$ being outer in $H^p(A^*) = (H^p)^*$. The latter is equivalent to $(H^p)^* = [h^*A^*]_p$. Taking adjoints again gives the result. □

**Remark.** The last result has the consequence that the theory has a left-right symmetry; for example our inner-outer factorizations $f = uh$ below may instead be done with $f = hu$ (a different $u, h$ of course).

Another classical theorem of Riesz-Szegö states that if $f \in L^1$ with $f \geq 0$, then $\int \log f > -\infty$ iff $f = |h|$ for an outer $h \in H^1$ iff $f = |h|^2$ for an outer $h \in H^2$. One may easily generalize such classical results if the algebra $A$ is antisymmetric.

THEOREM 7.7. *If $f \in L^1(M)_+$ and if $A$ is antisymmetric then the following are equivalent:*

(i) $\Delta(f) > 0$,
(ii) $f \notin [fA_0]_1$,
(iii) $f = |h|$ *for $h$ outer in $H^1$.*

PROOF. (i) $\Leftrightarrow$ (ii) See Corollary 6.4.

(ii) $\Rightarrow$ (iii) By the BN-factorization 4.2 (b), $f = uh$ for a unitary $u$ and an outer $h \in H^1$. Since $f \geq 0$ we have $f = (f^*f)^{\frac{1}{2}} = |h|$.

(iii) $\Rightarrow$ (ii) Conversely, if $h$ is outer then by the Lemma 7.1 we have $h \notin [hA_0]_1$. By Lemma 7.2 it follows that $|h| \notin [|h|A_0]_1$. □

We now turn to the topic of 'inner-outer' factorizations. It can be shown as in the classical case that if $f = uh$ is a factorization' of an $f \in L^p(M)$, for a unitary $u \in M$ and $h$ outer in $H^1$, then this factorization is unique up to a unitary in $\mathcal{D}$. Namely, if $u_1h_1 = u_2h_2$ were two such factorizations, then $u_2 = u_1u$ and $h_2 = u^*h_1$. See [**10**] for the easy details.

THEOREM 7.8. *If $f \in L^1(M)$, if $1 \leq p < \infty$, and if $A$ is antisymmetric, then the following are equivalent:*

(i) $\Delta(f) > 0$,
(ii) $f \notin [fA_0]_1$,
(iii) $|f|^{\frac{1}{p}} \notin [|f|^{\frac{1}{p}}A_0]_p$,
(iv) $f = uh$ *for a unitary $u \in M$ and $h$ outer in $H^1$.*

*The factorization in* (iv) *is unique up to a unimodular constant.*

PROOF. (i) $\Rightarrow$ (iv) If $f \in L^1(M)$ with $\Delta(f) > 0$, then $\Delta(|f|) > 0$, and so $|f| = |h|$ with $h$ outer in $H^1$, by the previous result. It follows by Lemma 7.2 that $f = uvh$ for a partial isometry $u$ and a unitary $v$ in $M$. We have $\Delta(f) = \Delta(u)\Delta(v)\Delta(h)$ by Theorem 6.1 (4), so that $\Delta(u) > 0$. Thus $\Delta(u^*u) > 0$. This forces $u^*u = 1$ by [**2**, p. 606], and so $u$ is unitary.

(iii) $\Leftrightarrow$ (i) See Corollary 6.4.

(iv) $\Rightarrow$ (ii) If $f \in [fA_0]_1$, then $h \in [hA_0]_1$, and by Lemma 7.1 we obtain a contradiction.

(ii) $\Rightarrow$ (iv) This is the BN-factorization 4.2 (b).

(iv) $\Rightarrow$ (i) We have $\Delta(uh) = \Delta(u)\Delta(h) = \Delta(h) > 0$, by previous results.

The uniqueness assertion follows from the remark above the theorem. □

**Remarks.** 1) The $u$ in (iv) is necessarily in $[fA]_1$ (indeed if $ha_n \to 1$ with $a_n \in A$, then $fa_n = uha_n \to u$).

2) Suppose that in Theorem 7.8, $f$ is also in $H^1$. Then, first, the $u$ in (iv) is necessarily in $[fA]_1 \subset H^1$, by Remark 1. So $u \in H^1 \cap M = A$ (using $\tau$-maximality for example). Thus $u$ is 'inner' (i.e. is a unitary in $H^\infty = A$). Second, note that (i)–(iv) will hold if $\tau(f) \neq 0$. Indeed if $f \in H^1$ and if $f \in [fA_0]_1$ with $fa_n \to f$ for $a_n \in A_0$, then $\tau(f) = \lim_n \tau(fa_n) = 0$.

COROLLARY 7.9. *If $A$ is antisymmetric and $f \in L^1(M)_+$ then $\Delta(f) > 0$ iff $f = |h|^p$ for an outer $h \in H^p$.*

Proof. ($\Rightarrow$) By the previous result, $f^{\frac{1}{p}} \notin [f^{\frac{1}{p}} A_0]_p$, and so by the Beurling-Nevanlinna factorization 4.2 (b) we have $f^{\frac{1}{p}} = uh$, where $h$ is outer in $H^p$, and $u$ is unitary. Thus $f = (f^{\frac{1}{p}} f^{\frac{1}{p}})^{\frac{p}{2}} = (h^* h)^{\frac{p}{2}} = |h|^p$.

($\Leftarrow$) If $f = |h|^p$ for an outer $h \in H^p$ then $\Delta(f) = \Delta(|h|)^p > 0$ by Theorem 6.1 (3) and Theorem 7.4. □

Corollary 7.10. *If $A$ is antisymmetric and $f \in L^p(M)$ then $\Delta(f) > 0$ iff $f = uh$ for a unitary $u$ and an outer $h \in H^p$. If $p = 2$ and $v$ is the orthogonal projection of $f$ onto $[fA_0]_2$, then $u$ is the partial isometry in the polar decomposition of $k - v$, and $h = u^* k$.*

Proof. ($\Rightarrow$) By Theorem 7.8 we obtain the factorization with outer $h \in H^1$. Since $|f| = |h|$ we have $h \in L^p(M) \cap H^1 = H^p$ (using [**52**, Proposition 2]).

($\Leftarrow$) As in the proof that (iv) implied (i) in Theorem 7.8.

See [**10**] for the proof of the last part (this is not used below). □

An obvious question is whether there are larger classes of subalgebras of $M$ besides subdiagonal algebras for which such classical factorization theorems hold. The following shows that, with a qualification, the answer to this is in the negative. We omit the proof, which may be found in [**10**], and proceeds by showing that $A$ satisfies (ii) in Theorem 1.2 above.

Proposition 7.11. *Suppose that $A$ is a tracial subalgebra of $M$, such that every $f \in L^2(M)$ with $\Delta(f) > 0$ is a product $f = uh$ for a unitary $u$ and an outer $h \in [A]_2$. Then $A$ is a finite maximal subdiagonal algebra.*

*Question.* Is there a characterization of outers in $H^1$ in terms of extremals, as in the deLeeuw-Rudin theorem of e.g. [**24**, p. 139–142], or [**16**, pp. 137-139]?

## 8. Logmodularity, operator spaces, and the uniqueness of extensions

In the material above, we have not used operator spaces. This is not because they are not present, but rather because they are not necessary. However, the algebras above do have interesting operator space properties, and this would seem to add to their importance. For example:

Theorem 8.1. [**6, 11**] *Suppose that $A$ is a logmodular subalgebra of a unital $C^*$-algebra $B$, with $1_B \in A$.*

(1) *Any unital completely contractive (resp. completely isometric) homomorphism $\pi \colon A \to B(H)$ has a unique extension to a completely positive and completely contractive (resp. and completely isometric) map from $B$ into $B(H)$.*
(2) *Every $*$-representation of $B$ is a boundary representation for $A$ in the sense of* [**3**].

Proposition 8.2. *The $C^*$-envelope (or 'noncommutative Shilov boundary') of a maximal subdiagonal subalgebra of $M$ is $M$ again. If $M$ is also injective then the injective envelope of a maximal subdiagonal subalgebra of $M$ is $M$ again.*

Proof. The proof of the first assertion may be found in [**6, 11**]. The last statement follows from the earlier one, since $M = I(M) = I(C_e^*(A)) = I(A)$. The last equality is valid for any unital operator space $A$ by the 'rigidity' characterization of the injective envelope [**11**, Section 4.2], since $A \subset C_e^*(A) \subset I(A)$. □

There is a partial converse to some of the above:

THEOREM 8.3. [**9**] *Suppose that $A$ is a subalgebra of a unital $C^*$-algebra $B$ such that $1_B \in A$, and suppose that $A$ has the property that for every Hilbert space $H$, every completely contractive unital homomorphism $\pi : A \to B(H)$ has a unique completely contractive (or equiv. completely positive) extension $B \to B(H)$. Then $B = C_e^*(A)$, the $C^*$-envelope of $A$.*

*Question:* If $A$ is a tracial subalgebra of $M$ such that every completely contractive unital homomorphism $\pi : A \to B(H)$ has a unique completely contractive extension $B \to B(H)$, then is $A$ maximal subdiagonal? If this were true, it would be a noncommutative analogue of Lumer's result from [**36**].

**Closing remarks/further open questions.** 1) A question worthy of consideration is the extent to which the results surveyed above may be extended to the setting of type III von Neumann algebras. Although most results in Arveson's paper [**2**] are stated for subdiagonal subalgebras of von Neumann algebras with a faithful normal tracial state, he does also consider subalgebras of more general von Neumann algebras. It would be interesting if there was some way to extend some of our results to this context. See e.g. recent work of Xu [**60**]. However there are some major challenges to overcome if the full cycle of ideas presented above is to extend to the type III case. We will try to elaborate this point. In the preceding theory the Fuglede-Kadison determinant plays a vital role. One expects that for a type III theory to work, a comparable quantity would have to found in that context. The problem is that any von Neumann algebra which admits of a Fuglede-Kadison determinant *necessarily* admits of a faithful normal tracial state (see e.g. [**30**, Theorem 3.2]). Hence if the theory is to extend to the case of, say, a type III algebra $M$ equipped with a faithful normal state $\varphi$, then in order to have the necessary tools at hand, one would first have to establish a theory of a determinant-like quantity defined in terms of $\varphi$. Clearly such a quantity cannot be a proper Fuglede-Kadison determinant as such, but if it exists, one expects it to exhibit determinant-like behavior with respect to the canonical modular automorphism group induced by $\varphi$: a kind of modular-determinant. In support of the contention that such a quantity exists we note that at least locally $\varphi$ does induce a determinant on $M$. That is, any maximal abelian subalgebra of $M$ does admit of a Fuglede-Kadison determinant, since the restriction of $\varphi$ to such a subalgebra is trivially a faithful normal tracial state. The challenge is to find a quantity which will yield a global expression for this local behavior.

2) A nice question suggested by Gilles Godefroy at this conference is whether subdiagonal algebras have unique preduals. This question is very natural in the light of his positive result in this direction [**17**, Theoreme 33] in the function algebra case (generalizing Ando's classical result on the uniqueness of predual for $H^\infty(\mathbb{D})$).

3) Another interesting question suggested to us at this conference, by R. Rochberg, is whether the generalization of the Helson-Szegö theorem to weak* Dirichlet algebras [**23**], and its corollary on invertibility of Toeplitz operators, has a noncommutative variant in this setting. This would probably need to use the real variable methods developed by Marsalli and West in [**38**], as well as their theory of Toeplitz operators with noncommuting symbols developed in [**39**].

4) B. Wick has suggested to us that it might be worth investigating noncommutative analogues of [**51**, Theorem 18.18], which may be viewed as a simple 'uniform variant' of the corona theorem.

5) Can one characterize (complete) isometries between noncommutative $H^p$ spaces? We have been informed by Fleming and Jamison that there have been recent breakthroughs in the study of isometries between classical $H^p$ spaces. See [**32**] for some work in this direction in the noncommutative case.

6) It seems extremely worthwhile, and Arveson has also suggested to us, to investigate certain subdiagonal algebras (or algebras which are not far from being subdiagonal) coming from free group examples as in [**2**, Section 3] in the framework of current free probability theory. For example, it seems that aspects of our subject are not very far from some perspectives from the recent studies, by Haagerup and his collaborators, of the invariant subspace problem relative to a finite von Neumann algebra (see e.g. [**18, 19**]).

7) A most interesting project would be the widening of the class of algebras to which (parts of) the theory above may be extended. There may in fact be several directions in which to proceed. For example, as in development of the commutative theory (see e.g. [**16, 4**]), one could try to replace the requirement that $\Phi$ is multiplicative on $A$, by the multiplicativity of another conditional expectation $\Psi$. If $\Delta(\Psi(a)) \leq \Delta(a)$ for $a \in A$, then one could view $\Psi$ as a *noncommutative Jensen measure*, and develop a theory of the latter objects parallelling the important classical theory. Other clues to the enlarging of the class of algebras might come from item 6 above, or from a closer examination of known classes of noncommutative algebras which have some intersection with the class of subdiagonal algebras, and which do have noncommutative Hardy space properties. Such algebras have been studied in the 1970s and later, beginning with papers such as [**61**].

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