Lab 01/06

(*) $\sum_{k=2}^{\infty} \frac{1}{k^2+k}$

$$\frac{1}{k^2+k} = \frac{1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1}$$

$$= \frac{(k+1)-k}{k(k+1)}$$

$$= \frac{\cancel{(k+1)}}{k\cancel{(k+1)}} - \frac{\cancel{k}}{\cancel{k}(k+1)}$$

$$= \frac{1}{k} - \frac{1}{k+1}$$

$$= \sum_{k=2}^{\infty}\left[\frac{1}{k} - \frac{1}{k+1}\right]$$

$$= \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) - \cancel{+} - - - .$$

$$= \frac{1}{2}$$

(*) $-1 + 5 + 11 + 17 + \cdots + 59$

$\uparrow 0 \quad \uparrow 1 \quad \uparrow 2 \quad \uparrow 3$

$$\sum_{k=0}^{10} 6k-1$$

(*) $\sum_{k=1}^{\infty} \frac{\tan^{-1} 2k}{24k^2+6}$

$$\leq \sum_{k=1}^{\infty} \frac{\pi/2}{24k^2+6}$$

$$\leq \frac{\pi}{48} \sum \frac{1}{k^2}$$

$f(x) = \frac{\tan^{-1}(2x)}{24x^2+6}$

Integral test

$\sum \frac{1}{k^2}$ is conv by p-series test as $p = 2 > 1$

$\therefore$ By BCT $\sum \frac{\tan^{-1} 2k}{24k^2+6}$ is conv.

$$\sum \frac{\tan^{-1}2k}{24k^2+6} = \frac{1}{6}\sum \frac{\tan^{-1}2k}{4k^2+1}$$

$$f(x) = \frac{\tan^{-1}2x}{4x^2+1}$$

(i) $f(x)$ cts

(ii) $f(x)$ decreasing

(iii) $f(x)$ +ve

$$\int_1^\infty \frac{\tan^{-1}2x}{4x^2+1}\,dx$$

$$= \lim_{b\to\infty} \int_1^b \frac{\tan^{-1}2x}{4x^2+1}\,dx$$

$$\int \frac{\tan^{-1}2x}{4x^2+1}\,dx$$

$u = \tan^{-1}2x$

$du = \frac{1}{4x^2+1}\,dx$

$$= \int u\,du$$

$$= \frac{u^2}{2}+c = (\tan^{-1}2x)^2+c$$

$$\int_1^\infty \frac{\tan^{-1}2x}{1+4x^2}\,dx = \lim_{b\to\infty} (\tan^{-1}2x)^2\Big|_1^b$$

$$= \lim_{b\to\infty} (\tan^{-1}2b)^2 - (\tan^{-1}2)^2$$

$$= \frac{\pi}{2} - (\tan^{-1}2)^2 < \infty$$

$\therefore$ By integral test the series is conv.

(*) $\sum \frac{1}{3+2^{-k}}$

BDT
$a_k \not\to 0$ $\sum a_k$ is div

BDT does say any about conv

$a_k \to 0$
BDT X

$a_k = \frac{1}{3+2^{-k}}$

$= \frac{1}{3+\frac{1}{2^k}} \to \frac{1}{3}$ as $k \to \infty$

$\neq 0$

$\sum a_k$ is div

(*) $\sum \frac{\ln\sqrt{4+k}}{3k} = \frac{1}{6}\sum \frac{\ln(4+k)}{k} \geq \frac{1}{6}\sum \frac{\ln k}{k}$

$f(x) = \frac{\ln(x)}{x}$

(i) $f(x)$ +ve $x > \ln x$
(ii) $f(x)$ dec
(iii) $f(x)$ conts

$\int_1^\infty \frac{\ln(x)}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln(x)}{x}\,dx$

$\int \frac{\ln(x)}{x}\,dx$

$u = \ln x$
$du = \frac{1}{x}\,dx$

$= \int u\,du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C$

$\int_1^\infty \frac{\ln(x)}{x}\,dx = \lim_{b\to\infty} \frac{(\ln x)^2}{2}\Big|_1^b = \text{DNE}$

$\sum \frac{\ln(x)}{x}$ is div by integral test

(*) $\sum \frac{3ke^{-k^2}}{8} = \frac{3}{8} \sum \frac{k}{e^{k^2}}$

(i) $f(x)$ +ve (ii) $f(x)$ dec

(ii) $f(x)$ is cts

$f(x) = \frac{x}{e^{x^2}}$

$\int_1^\infty \frac{x}{e^{x^2}} dx = \lim_{b \to \infty} \int_1^b x e^{-x^2} dx$

$\int x e^{-x^2} dx$ $\quad u = x^2 \quad du = 2x\,dx$

$= \frac{1}{2} \int e^{-u} du$

$= \frac{1}{2} \frac{e^{-u}}{-1} + C = \frac{-e^{-u}}{2} + C = -\frac{e^{-x^2}}{2} + C$

$\int_1^\infty \frac{x}{e^{x^2}} dx = \lim_{b \to \infty} -\frac{e^{-x^2}}{2} \Big|_1^b = \lim_{b \to \infty} -\frac{e^{-b^2}}{2} + \frac{e^{-1}}{2}$

$= \frac{1}{2e} < \infty$

$\therefore$ By integral test the series is cnvgt

(*) $\sum \frac{3 + \cos k}{\sqrt{k+4}} \geq \sum \frac{2}{\sqrt{k+4}}$

$\geq \sum \frac{2}{\sqrt{2k}}$

$= \sum \frac{2}{\sqrt{2}\, k^{1/2}}$

$\frac{1}{2} = 0.5$

$\frac{1}{3} = 0.33$

$\frac{1}{4} = 0.25$

$k+4 < k+k$

$k+4 < 2k$

$\frac{1}{k+4} > \frac{1}{2k}$

$\sum \frac{1}{k^{1/2}}$ is div by p-series test $p = \frac{1}{2} < 1$

$\therefore$ The series is div by BCT

(*) $$\sum_{k=2}^{\infty} \frac{1-3^k}{5^k} = \sum_{k=2}^{\infty}\left(\frac{1}{5}\right)^k - \sum_{k=2}^{\infty}\left(\frac{3}{5}\right)^k$$

$a + ar + ar^2 + \cdots \cdots$

r - common ratio

$$\text{Sum} = \frac{a}{1-r}$$

$|r| < 1$

$r = \frac{1}{5}$

$|r| = \frac{1}{5} < 1$

$$\text{sum} = \frac{1/25}{1-\frac{1}{5}}$$

$$= \frac{1/25}{4/5} = \frac{1}{20}$$

$r = \frac{3}{5}$

$|r| = \frac{3}{5} < 1$

$$\text{Sum} = \frac{9/25}{1-\frac{3}{5}}$$

$$= \frac{9/25}{2/5}$$

$$= \frac{9}{10}$$

$$\text{Sum} = \frac{1}{20} - \frac{9}{10} = \frac{1}{20} - \frac{18}{20} = \frac{-17}{20}$$

$\circledast$ $\Sigma a_k \; \sum \dfrac{2k^3+3k-1}{5k^5-6k^3+7}$ $\qquad \Sigma b_k = \sum \dfrac{1}{k^2}$

$a_k = \dfrac{2k^3+3k-1}{5k^5-6k^3+7}$ $\qquad b_k = \dfrac{1}{k^2}$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{2k^5+3k^3-k^2}{5k^5-6k^3+7} = \frac{2}{5} > 0$$

This does not say anything about conv/div

Σb_k is conv by p-series test as $p = 2 > 1$

$\therefore$ Σa_k is conv by LCT

$\circledast$ $\displaystyle\sum \frac{1}{\sqrt{2k^2-3k}} = \Sigma a_k$ $\qquad \Sigma b_k = \sum \dfrac{1}{k}$

$$\lim_{k\to\infty} \frac{a_k}{a_k} = \lim_{k\to\infty} \frac{k}{\sqrt{2k^2-3k}} = \frac{1}{\sqrt{2}}$$

Σb_k is div by p series test $P = 1$

Σa_k is div by LCT

$\displaystyle\sum \frac{2k+3}{\sqrt{3k^5-2k^3}} = \Sigma a_k$ $\qquad \Sigma b_k = \sum \dfrac{1}{k^{3/2}}$

(*) $\sum \frac{2+\cos k}{1+k^2} \leq \sum \frac{3}{1+k^2} < \sum \frac{3}{k^2}$

$\sum \frac{1}{k^2}$ is conv by p-seires test $p=2>1$

$\sum \frac{2+\cos k}{1+k^2}$ is conv by DCT

(*) $\sum \left(\frac{2k}{5k+2}\right)^k$ $\quad a_k = \left(\frac{2k}{5k+2}\right)^k$

$$\lim_{k\to\infty} (a_k)^{1/k} = \lim_{k\to\infty} \left(\left(\frac{2k}{5k+2}\right)^k \right)^{1/k}$$

$$= \lim_{k\to\infty} \frac{2k}{5k+2} = \frac{2}{5} < 1$$

convgt by Root test

(*) $\sum \left(\frac{k}{k+3}\right)^k$

$$a_k = \left(\frac{k}{k+3}\right)^k$$

$$= \left(\frac{k+3}{k}\right)^{-k}$$

$$= \left(1+\frac{3}{k}\right)^{-k}$$

$$\to e^{-3} \neq 0 \quad \text{as } k\to\infty$$

$\therefore$ Div by BDT

(*) $\sum \frac{3^k (k+1)!}{k^k}$

$$a_k = \frac{3^k (k+1)!}{k^k}$$

$$\frac{a_{k+1}}{a_k} = \frac{3^{k+1}(k+2)!}{(k+1)^{k+1}} \frac{k^k}{3^k (k+1)!}$$

$$= \frac{3^{k+1}}{3^k} \frac{k^k}{(k+1)^{k+1}} \frac{(k+2)!}{(k+1)!}$$

$$= \frac{\cancel{3^k}.3}{\cancel{3^k}} \frac{k^k}{(k+1)^k (k+1)} (k+2)$$

$$= 3 \left(\frac{k}{k+1}\right)^k \left(\frac{k+2}{k+1}\right) \longrightarrow 3 \text{ as } k \to \infty$$

$$> 1$$

Div by Ratio test

(*) $\sum \frac{(k!)^2}{(2k)!}$ $\quad a_k = \frac{(k!)^2}{(2k)!}$

$$\frac{a_{k+1}}{a_k} = \frac{((k+1)!)^2}{(2k+2)!} \frac{(2k)!}{(k!)^2}$$

$$= \left(\frac{(k+1)!}{k!}\right)^2 \frac{(2k)!}{(2k+2)!}$$

$$\frac{k^2+1+2k}{4k^2+6k+2} = \frac{(k+1)^2}{(2k+1)(2k+2)} \to \frac{1}{4} < 1 \text{ as } k \to \infty$$

Conv by Ratio test

(*) $\sum \frac{9 \cdot 18 \cdot 27 \cdots (9k)}{(9k)!}$

$$= \sum \frac{(9 \cdot 1)(9 \cdot 2)(9 \cdot 3) \cdots (9 \cdot k)}{(9k)!}$$

$$= \sum \frac{9^k (1 \cdot 2 \cdot 3 \cdots k)}{(9k)!} = \sum \frac{9^k k!}{(9k)!}$$

(*) $f(x) = x^3 e^{x/3}$ $\qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$= x^3 \sum_{n=0}^{\infty} \frac{\left(\frac{x}{3}\right)^n}{n!}$$

$$= x^3 \sum \frac{x^n}{3^n n!} = \sum_{n=0}^{\infty} \frac{x^{n+3}}{3^n n!}$$

$$f(x) = x^3 + \frac{x^4}{3} + \frac{x^5}{3^2 \cdot 2!} + \frac{x^6}{3^3 3!} + \frac{x^7}{3^4 4!} + \cdots$$

$x^3 \to \underset{1}{3x^2} \to \underset{2}{6x} \to \underset{3}{6} \to \underset{4}{0} \to \underset{5}{0} \to \underset{6}{0}$

$x^4 \to \underset{1}{4x^3} \to \underset{2}{12x^2} \to \underset{3}{24x} \to \underset{4}{24} \to \underset{5^{th}}{0} \to \underset{6^{th}}{0}$

x^5 - - - - . . $\underset{6^{th}}{0}$

x^6 — — — - . . . - - . $\underset{6^{th}}{6!}$

x^7 — — — — — — - - . . $7!\,x_{\,6^{th}}$

$$f^6(x) = \frac{6!}{3^3 3!} + \frac{7!\,x}{3^4 4!} + \frac{\frac{8!}{2}x^2}{3^5 5!} + \cdots$$

$$f^6(0) = \frac{6!}{3^3 3!} = \frac{\cancel{720}\ \cancel{120}\ 40}{\cancel{27}_{9} \cdot \cancel{6}} = \frac{40}{9}$$