Math 3321

Homogeneous Equations with Constant Coefficients

University of Houston

Lecture 08

Outline

Homogeneous Equations with Constant Coefficients

In the last lecture, we stated that there is no general method for solving second (or higher) order linear differential equations. In this section we will be looking at a special case where we will be able to establish a method for finding solutions.

Definition

A *second order linear homogeneous differential equation with constant coefficients* is an equation which can be written in the form

$$y'' + ay' + by = 0. \tag{1}$$

Where a and b are real numbers.

Homogeneous Equations with Constant Coefficients

Consider the differential equation below which can be viewed as either first order linear or separable.

$$y' + ay = 0$$

We can solve this to find the general solution is

$$y = Ce^{-ax}.$$

This leads us to a guess at the structure of solutions to (1). That is, we might be able to find solutions of the form $y = e^{rx}$. When $y = e^{rx}$, we have $y' = re^{rx}$ and $y'' = r^2e^{rx}$. Putting this into (1), we get:

$$0 = y'' + ay' + by = r^2e^{rx} + are^{rx} + be^{rx} = e^{rx}[r^2 + ar + b].$$

This equation holds if and only if

$$r^2 + ar + b = 0.$$

Homogeneous Equations with Constant Coefficients

Definitions

Given the differential equation (1), the corresponding quadratic equation

$$r^2 + ar + b = 0 \tag{2}$$

is called the *characteristic equation* of (1). The quadratic polynomial $r^2 + ar + b$ is the *characteristic polynomial*. Finally, the roots of the equation/polynomial are known as the *characteristic roots*.

Solution Cases

Roots of the Characteristic Equation

The look of our solution functions will depend on the roots of the characteristic equation. We have three cases:

1. Equation (2) has two distinct real roots, $r_1 = \alpha$, $r_2 = \beta$.
2. Equation (2) has only one repeated root, $r = \alpha$.
3. Equation (2) has complex congjugate roots, $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$.

Solution Cases

Case 1: Two Real Roots

When we have two real roots for the characterstic equation, $r_1 = \alpha$, $r_2 = \beta$, we get the solutions

$$y_1(x) = e^{\alpha x} \text{ and } y_2(x) = e^{\beta x}$$

for (2). Since $\alpha \neq \beta$, y_1 and y_2 are not constant multiples of each other and $\{y_1, y_2\}$ will constitute a fundamental set of solutions for equation (1) so that the general solution is

$$y = C_1 e^{\alpha x} + C_2 e^{\beta x}.$$

Solution Cases

Examples:

1. Find the general solution of the given differential equation.

$$y'' + 2y' - 8y = 0$$

Solution Cases

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$$y'' + 2y' - 8y = 0$$

We write the characteristic equation

$$r^2 + 2r - 8 = 0$$

The roots are $r_1 = -4, r_2 = 2$

We are in the situation of 2 distinct real roots.
Thus the general solution is

$$y(x) = c_1 e^{-4x} + c_2 e^{2x}$$

Solution Cases

2. Find two linearly independent solutions and the general solution of the given differential equation.

$$y'' - 4y' = 0$$

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$$y'' - 4y' = 0$$

We write the characteristic equation

$$r^2 - 4 = 0$$

The roots are $r_1 = 2, r_2 = -2$

We are in the situation of 2 distinct real roots.
Thus the general solution is

$$y(x) = c_1 e^{2x} + c_2 e^{-2x}$$

Solution Cases

Case 2: One Repeated Real Root

Here the characteristic equation has a single repeated root, $r = \alpha$. In this case,

$$y_1(x) = e^{\alpha x} \text{ and } y_2(x) = xe^{\alpha x}$$

are linearly independent solutions for (2) so that the general solution is

$$y = C_1 e^{\alpha x} + C_2 x e^{\alpha x}.$$

Solution Cases

3. Find a fundamental set of solutions of the given differential equation.

$$y'' + 2y' + y = 0$$

3. Find a fundamental set of solutions of the given differential equation.

$$y'' + 2y' + y = 0$$

We write the characteristic equation

$$r^2 + 2r + 1 = 0$$

The roots are $r_1 = r_2 = -1$

We are in the situation of 2 repeated real roots.
Thus the general solution is

$$y(x) = c_1 e^{-x} + c_2 x e^{-x}$$

Solution Cases

4. Find the general solution of the given differential equation.

$$y'' - 4y' + 4y = 0$$

Solution Cases

4. Find the general solution of the given differential equation.

$$y'' - 4y' + 4y = 0$$

We write the characteristic equation

$$r^2 - 4r + 4 = 0$$

The roots are $r_1 = r_2 = 2$

We are in the situation of 2 repeated real roots.
Thus the general solution is

$$y(x) = c_1 e^{2x} + c_2 x e^{2x}$$

Solution Cases

Case 3: Complex Conjugate Roots

The characteristic equation has a pair of complex conjugate roots,

$$r_1 = \alpha + i\beta,\ r_2 = \alpha - i\beta,\ \text{where } \beta \neq 0.$$

In this case,

$$y_1(x) = e^{\alpha x}\cos(\beta x) \text{ and } y_2(x) = e^{\alpha x}\sin(\beta x)$$

are linearly independent solutions for (2) so that the general solution is

$$y = C_1 e^{\alpha x}\cos(\beta x) + C_2 e^{\alpha x}\sin(\beta x).$$

Solution Cases

Euler's Formula

The solutions from Case 3 above involve the use of *Euler's formula* which says

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$

where $i = \sqrt{-1}$ is the imaginary unit.

Hence

$$e^{\alpha+i\beta} = e^{\alpha}e^{i\beta} = e^{\alpha}(\cos(\beta) + i\sin(\beta))$$

$$e^{\alpha-i\beta} = e^{\alpha}e^{-i\beta} = e^{\alpha}(\cos(\beta) - i\sin(\beta))$$

A linear combination of $e^{\alpha+i\beta}$ and $e^{\alpha+i\beta}$ is equivalent to a linear combination of $e^{\alpha}\cos(\beta)$ and $e^{\alpha}\sin(\beta)$

Solution Cases

5. Find a fundamental set of solutions of the given differential equation.

$$y'' + 2y' + 17y = 0$$

Solution Cases

5. Find a fundamental set of solutions of the given differential equation.
$$y'' + 2y' + 17y = 0$$

We write the characteristic equation

$$r^2 + 2r + 17 = 0$$

The roots are

$$r_1, r_2 = \frac{-2 \pm \sqrt{4 - (4)(17)}}{2} = \frac{-2 \pm i\sqrt{6}4}{2} = -1 \pm i4$$

We are in the situation of complex conjugate roots.
Thus the general solution is

$$y(x) = c_1 e^{-x} \cos(4x) + c_2 e^{-x} \sin(4x)$$

Solution Cases

Important Special Case

Consider the differential equation $y'' + by = 0,\ b > 0$. Let $\lambda = \sqrt{b}$ and realize the equation is then

$$y'' + \lambda^2 y = 0$$

with characteristic equation $r^2 + \lambda^2 = 0$. We find the roots to be $r = \pm\lambda i$ so that two linearly independent solutions of the differential equation are

$$y_1(x) = \cos(\lambda x) \text{ and } y_2(x) = \sin(\lambda x)$$

and the general solution is

$$y = C_1 \cos(\lambda x) + C_2 \sin(\lambda x).$$

Solution Cases

6. Find the general solution of the given differential equation.

$$y'' + 4y = 0$$

Solution Cases

6. Find the general solution of the given differential equation.

$$y'' + 4y = 0$$

We write the characteristic equation

$$r^2 + 4 = 0$$

The roots are

$$r_1, r_2 = \sqrt{-4} = \pm i2$$

We are in the situation of complex conjugate roots, with no real part
Thus the general solution is

$$y(x) = c_1 \cos(2x) + c_2 \sin(2x)$$

Recovering a Differential Equation from Solutions

Examples:

1. Find a second order, linear, homogeneous differential equation with constant coefficients that has the functions $y_1 = e^x$ and $y_2 = e^{-3x}$ as solutions.

Recovering a Differential Equation from Solutions

Examples:

1. Find a second order, linear, homogeneous differential equation with constant coefficients that has the functions $y_1 = e^x$ and $y_2 = e^{-3x}$ as solutions.

We need to write the characteristic equation under the assumption that we have two distinct real roots $r_1 = 1$, $r_2 = -3$

$$(r-1)(r+3) = r^2 + 2r - 3 = 0$$

Hence the characteristic polynomial is

$$p(r) = r^2 + 2r - 3$$

and the Differential Equation is

$$y'' + 2y' - 3y = 0$$

2. Find a second order, linear, homogeneous differential equation with constant coefficients that has $y = C_1 + C_2 e^{-4x}$ as its general solution.

Recovering a Differential Equation from Solutions

2. Find a second order, linear, homogeneous differential equation with constant coefficients that has $y = C_1 + C_2 e^{-4x}$ as its general solution.

We need to write the characteristic equation under the assumption that we have two distinct real roots $r_1 = 0$, $r_2 = -4$

$$(r)(r+4) = r^2 + 4r = 0$$

Hence the Differential Equation is

$$y'' + 4y' = 0$$

Recovering a Differential Equation from Solutions

3. Find a second order, linear, homogeneous differential equation with constant coefficients that has $y = xe^{2x}$ as a solution.

Recovering a Differential Equation from Solutions

3. Find a second order, linear, homogeneous differential equation with constant coefficients that has $y = xe^{2x}$ as a solution.

We need to write the characteristic equation under the assumption that we have two repeated real roots $r_1 = r_2 = 2$. This is the only situation where we have a solution of the form $y = xe^{2x}$.

$$(r-2)^2 = r^2 - 2r + 4 = 0$$

Hence the Differential Equation is

$$y'' - 2y' + 4y = 0$$

The general solution of the ODE is

$$y(x) = c_1 e^{2x} + c_2 x e^{2x}$$

Recovering a Differential Equation from Solutions

4. Find a second order, linear, homogeneous differential equation with constant coefficients that has $y = 5e^{2x}\sin(3x)$ as a solution.

Recovering a Differential Equation from Solutions

4. Find a second order, linear, homogeneous differential equation with constant coefficients that has $y = 5e^{2x}\sin(3x)$ as a solution.

We need to write the characteristic equation under the assumption that we have complex conjugate roots $r = 2 \pm i3$. This is the only situation where we have a solution of the form $y = 5e^{2x}\sin(3x)$.

$$\begin{aligned}(r-2-3i)(r-2+3i) &= (r-2)^2 - (i3)^2 \\ &= r^2 - 4r + 4 - (-9) \\ &= r^2 - 4r + 13 = 0\end{aligned}$$

Hence the Differential Equation is

$$y'' - 4y' + 13y = 0$$

The general solution of the ODE is

$$y(x) = c_1 e^{2x}\cos(3x) + c_2 e^{2x}\sin(3x)$$