Math 3321

Systems of Linear Equations. Part I

University of Houston

Lecture 17

Outline

Introduction. Linear Equations

There is an important relation between differential equations and linear algebra.
As we will see later on, this relation is very important to derive methods to solve linear differential equations of high order.

Definition

A **linear (algebraic) equation in n unknowns**, $x_1, x_2, \ldots, x_n$, is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ and b are real numbers.

Special case, for $n = 1$, we have the linear equation

$$ax = b$$

Introduction. Linear Equations

We have the following observation

Fact

Given a linear equation $ax = b$, exactly one of following holds:
(1) there is precisely one solution (a unique solution)

$$x = a^{-1}b = \frac{b}{a}, \quad a \neq 0,$$

(2) there are no solutions

$$0x = b, \quad b \neq 0$$

(3) there are infinitely many solutions

$$0x = 0.$$

Introduction. Linear Equations

A solution of the equation

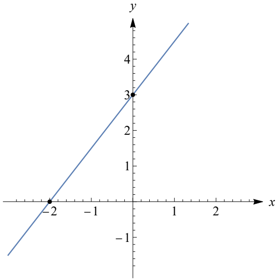
$$a_1 x + a_2 y = b$$

is an ordered pair of numbers (x_0, y_0).

Assuming a_1, a_2, not both 0, then the set of all ordered pairs that satisfy the equation is a straight line (in the x, y-plane). The equation has infinitely many solutions – the set of points that lie on the line.

Example:

$-3x + 2y = 6$



Introduction. Linear Equations

Consider the following system of two linear equations in two unknowns:

$$a_1 x + a_2 y = b_1$$

$$a_3 x + a_4 y = b_2$$

Can we find ordered pairs (x, y) that satisfy both equations simultaneously?

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Can we find ordered pairs (x, y) that satisfy both equations simultaneously?

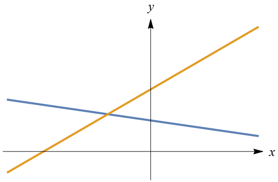
The answer is similar to the case of a linear equation with one unknown.

There are 3 possible cases: a unique solution, no solutions, infinitely many solutions.

To explain this situation, we can use the geometric intuition derived from the fact that each linear equation above is associated with an infinite line.

Introduction. Linear Equations

Two lines in the plane either
(a) have a unique point of intersection (the lines have different slopes), and the system has a unique solution



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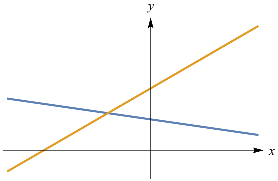
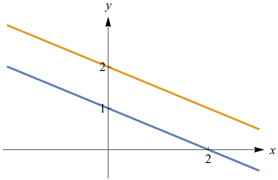
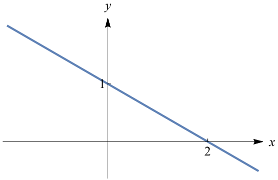


(b) are parallel (the lines have the same slope but, for example, different y-intercepts)
The system has NO solutions, there is no point that lies on <u>both</u> lines



(c) coincide (same slope, same y-intercept), and the system has infinitely many solutions.



An example for this case is

$$x + 2y = 2$$
$$2x + 4y = 4$$

Notice that the two equations are linearly dependent.

Introduction. Linear Equations

In summary:

Fact

Given a system of two linear equations in two unknowns

$$a_1x + a_2y = b_1$$
$$a_3x + a_4y = b_2$$

exactly one of following holds:
(a) **a unique solution**,
(b) **no solution**,
(c) **infinitely many solutions**.

Introduction. Linear Equations

Let us next consider a system of three linear equations in two unknowns:

$$a_1x + a_2y = b_1$$
$$a_3x + a_4y = b_2$$
$$a_5x + a_6y = b_3$$

Can we find ordered pairs (x, y) that satisfy the three equations simultaneously?

Introduction. Linear Equations

Let us next consider a system of three linear equations in two unknowns:

$$
\begin{aligned}
a_1x + a_2y &= b_1 \\
a_3x + a_4y &= b_2 \\
a_5x + a_6y &= b_3
\end{aligned}
$$

Can we find ordered pairs (x, y) that satisfy the three equations simultaneously?

We can again use the geometric intuition derived from the fact that each linear equation above is associated with an infinite line. Hence, exactly one of following holds:

(a) **a unique solution**,
(b) **no solution**,
(c) **infinitely many solutions**

Introduction. Linear Equations

Let us next consider a linear equations in three unknowns:

$$a_1x + a_2y + a_3z = b.$$

A solution of the equation is an ordered triple of numbers (x_0, y_0, z_0).

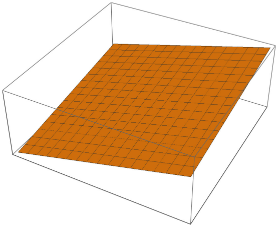
Introduction. Linear Equations

Let us next consider a linear equations in three unknowns:

$$a_1x + a_2y + a_3z = b.$$

A solution of the equation is an ordered triple of numbers (x_0, y_0, z_0).

Assuming a_1, a_2, a_3 are not all 0, then the set of all ordered triples that satisfy the equation is a plane



Introduction. Linear Equations

Consider a system of two linear equations in three unknowns

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

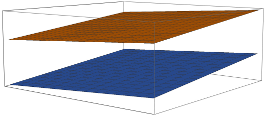
$$a_{21}x + a_{22}y + a_{23}z = b_2$$

Using the geometrical intuition, the solution set is the set associated with the possible intersection of the two planes.

Introduction. Linear Equations

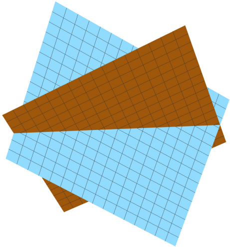
Hence, exactly one of following holds:

• Either the two planes are parallel (the **system has no solutions**),



• they coincide (**infinitely many solutions**, a whole plane of solutions),

• they intersect in a straight line (again, **infinitely many solutions**).



Introduction. Linear Equations

A system of three linear equations in three unknowns.

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

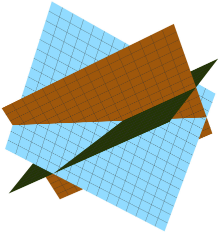
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

represents three planes in 3-space.

Using the geometrical intuition again, the solution set is the set associated with the possible intersection of the three planes.

Hence, exactly one of following holds:

(a) The system has a **unique solution**; the three planes have a unique point of intersection;



(b) The system has **infinitely many solutions**; the three planes intersect in a line, or the three planes intersect in a plane.

(c) The system has **no solution**; there is no point the lies on all three planes.

Solution of Systems of Linear Equations

Example 1: Solve the system

$$x + 3y = -5$$
$$2x - y = 4$$

Equivalent system (subtract from the second equation the first equation multiplied by 2)

$$x + 3y = -5$$
$$y = -2$$

Solution set:

$$x = 1, \quad y = -2$$

You can verify that this solution set satisfies both systems of equations

Solution of Systems of Linear Equations

Definition

Two systems of linear equations S_1 and S_2 are **equivalent** if they have exactly the same solution set.

Solution of Systems of Linear Equations

Definition

Two systems of linear equations S_1 and S_2 are **equivalent** if they have exactly the same solution set.

The operations that produce equivalent systems are called **elementary operations**.

1. Multiply/divide an equation by a nonzero number.
2. Interchange two equations.
3. Multiply an equation by a number and add it to another equation.

Solution of Systems of Linear Equations

Example 2: Solve the system

$$\begin{aligned} x - 2y + 4z &= 12 \\ 2x - y + 5z &= 18 \\ -x + 3y - 3z &= -8 \end{aligned}$$

By applying elementary operations, we obtain the equivalent system

$$\begin{aligned} x - 2y + 4z &= 12 \\ y - z &= -2 \\ z &= 3 \end{aligned}$$

Solution set:

$$x = 2, \quad y = 1, \quad z = 3$$

Solution of Systems of Linear Equations

Notice: it is much easier to compute the set from the second system

$$\begin{aligned} x - 2y + 4z &= 12 \\ y - z &= -2 \\ z &= 3 \end{aligned}$$

than the first one.

You use $z = 3$ into the second equation

$$y - 3 = -2$$

to find $y = 1$.
Next you use $z = 3$ and $y = 1$ into the first equation

$$x - 2 + 12 = 12$$

to find $x = 2$.

Example 3: Solve the system

$$\begin{aligned} 3x - 4y - z &= 3 \\ 2x - 3y + z &= 1 \\ x - 2y + 3z &= 2 \end{aligned}$$

Change the order of equation

$$\begin{aligned} x - 2y + 3z &= 2 \\ 3x - 4y - z &= 3 \\ 2x - 3y + z &= 1 \end{aligned}$$

row2 $\mapsto$ (row2 - 3 row1) and row3 $\mapsto$ (row3 - 2 row1)

$$\begin{aligned} x - 2y + 3z &= 2 \\ 2y - 10z &= -3 \\ y - 5z &= -3 \end{aligned}$$

Solution of Systems of Linear Equations

After interchanging row2 and row3

$$
\begin{aligned}
x - 2y + 3z &= 2 \\
y - 5z &= -3 \\
2y - 10z &= -3
\end{aligned}
$$

row3 $\mapsto$ (row3 - 2 row2)

$$
\begin{aligned}
x - 2y + 3z &= 2 \\
y - 5z &= -3 \\
01
\end{aligned}
$$

This shows that the system has no solution.

Solution of Systems of Linear Equations

Example 4: Solve the system

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= -2 \\ -2x_1 + 5x_2 - x_3 + 4x_4 &= 1 \\ 3x_1 - 7x_2 + 2x_3 + x_4 &= 9 \end{aligned}$$

row2 $\mapsto$ (row2 + 2 row1) and row3 $\mapsto$ (row3 - 3 row1)

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= -2 \\ x_2 + x_3 + 2x_4 &= -3 \\ -x_2 - x_3 + 4x_4 &= 15 \end{aligned}$$

row3 $\mapsto$ (row3 + row2)

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= -2 \\ x_2 + x_3 + 2x_4 &= -3 \\ 6x_4 &= 12 \end{aligned}$$

Solution of Systems of Linear Equations

Simplifying the last equation we get

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= -2 \\ x_2 + x_3 + 2x_4 &= -3 \\ x_4 &= 2 \end{aligned}$$

This gives the solution set:

$$\begin{aligned} x_1 &= -14 - 3a, \\ x_2 &= -7 - a, \\ x_3 &= a, \\ x_4 &= 2, \end{aligned}$$

where a is *any real number*.

The solution set contains infinitely many solutions

Solution of Systems of Linear Equations

Example 5: Solve the system

$$\begin{aligned} x - 2y + 4z &= 12 \\ 2x - y + 5z &= 18 \\ -x + 3y - 3z &= -8 \end{aligned}$$

To compute an equivalent **triangular** system, we need to manipulate the linear equations using elementary operations.

As you have noticed, these operations only affect the coefficients. Hence, we write below the **augmented matrix** of the system

$$\left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 2 & -1 & 5 & 18 \\ -1 & 3 & -3 & -8 \end{array}\right)$$

Solution of Systems of Linear Equations

r2 $\mapsto$ (r2 - 2 r1), r3 $\mapsto$ (r3 +r1)

$$\left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 2 & -1 & 5 & 18 \\ -1 & 3 & -3 & -8 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 0 & 3 & -3 & -6 \\ 0 & 1 & 1 & 4 \end{array}\right)$$

r3 $\mapsto \frac{1}{3}$ r3;   next r3 $\mapsto$ (r3 - r2)

$$\rightarrow \left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & 1 & 4 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 6 \end{array}\right)$$

r3 $\mapsto \frac{1}{2}$ r3

$$\rightarrow \left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Solution of Systems of Linear Equations

The triangular augmented matrix

$$\left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

corresponds to the (equivalent) system of equations:

$$\begin{aligned} x - 2y + 4z &= 12 \\ y - z &= -2 \\ z &= 3 \end{aligned}$$

Hence we obtain the solution set:

$$x = 2,\ y = 1,\ z = 3$$