Linear Algebra Review
and
Matlab Tutorial

Assigned Reading:

•Eero Simoncelli "A Geometric View of Linear Algebra"
http://www.cns.nyu.edu/~eero/NOTES/geomLinAlg.pdf

Background Material

- A computer vision "encyclopedia": CVonline.

  http://homepages.inf.ed.ac.uk/rbf/CVonline/

- Linear Algebra:
  - Eero Simoncelli "A Geometric View of Linear Algebra"
    http://www.cns.nyu.edu/~eero/NOTES/geomLinAlg.pdf
  - Michael Jordan slightly more in depth linear algebra review
    http://www.cs.brown.edu/courses/cs143/Materials/linalg_jordan_86.pdf
  - Online Introductory Linear Algebra Book by Jim Hefferon.
    http://joshua.smcvt.edu/linearalgebra/

Notation

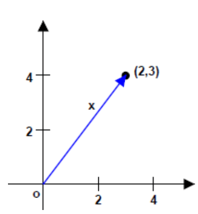
- Standard math textbook notation
  - **Scalars** are italic times roman: n, N
  - **Vectors** are bold lowercase: $\mathbf{x}$
    - Row vectors are denoted with a transpose: $\mathbf{x}^T$
  - **Matrices** are bold uppercase: $\mathbf{M}$
  - **Tensors** are calligraphic letters: $\mathcal{T}$

Overview

- Vectors in $\mathbb{R}^2$
- Scalar product
- Outer Product
- Bases and transformations
- Inverse Transformations
- Eigendecomposition
- Singular Value Decomposition

Warm-up: Vectors in $\mathbb{R}^n$

- We can think of vectors in two ways:
  - Points in a multidimensional space with respect to some coordinate system
  - translation of a point in a multidimensional space
    ex., translation of the origin (0,0)



Vectors in $\mathbb{R}^n$

- Notation:

$$\mathbf{x} \in \mathbb{R}^n, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathsf{T}}$$

- Length of a vector:

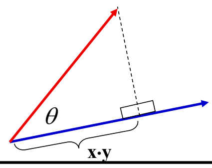
$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Dot product or scalar product

- Dot product is the product of two vectors
- Example:

$$\mathbf{x}\cdot\mathbf{y}=\begin{bmatrix}x_1\\x_2\end{bmatrix}\cdot\begin{bmatrix}y_1\\y_2\end{bmatrix}=x_1y_1+x_2y_2=s$$

- It is the projection of one vector onto another



$$\mathbf{x}\cdot\mathbf{y}=\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta$$

Scalar Product

- Notation

$$\langle\mathbf{x},\mathbf{y}\rangle$$

$$\mathbf{x}\cdot\mathbf{y}=\mathbf{x}^T\mathbf{y}=\begin{bmatrix}x_1 & \cdots & x_n\end{bmatrix}\begin{bmatrix}y_1\\\vdots\\y_n\end{bmatrix}$$

- We will use the last two notations to denote the dot product

Scalar Product

- Commutative: $\mathbf{x}\cdot\mathbf{y}=\mathbf{y}\cdot\mathbf{x}$
- Distributive: $(\mathbf{x}+\mathbf{y})\cdot\mathbf{z}=\mathbf{x}\cdot\mathbf{z}+\mathbf{y}\cdot\mathbf{z}$
- Linearity

$$(c\mathbf{x})\cdot\mathbf{y}=c(\mathbf{x}\cdot\mathbf{y})$$
$$\mathbf{x}\cdot(c\mathbf{y})=c(\mathbf{x}\cdot\mathbf{y})$$

$$(c_1\mathbf{x})\cdot(c_2\mathbf{y})=(c_1c_2)(\mathbf{x}\cdot\mathbf{y})$$

- Non-negativity:

$$\forall\mathbf{x}\neq 0:\ \langle\mathbf{x},\mathbf{x}\rangle>0 \qquad \langle\mathbf{x},\mathbf{x}\rangle=0\Leftrightarrow\mathbf{x}=0$$

- Orthogonality:

$$\forall\mathbf{x}\neq 0,\mathbf{y}\neq 0\quad \mathbf{x}\cdot\mathbf{y}=0\Leftrightarrow\mathbf{x}\perp\mathbf{y}$$

Norms in $\mathbb{R}^n$

- Euclidean norm (sometimes called 2-norm):

$$\|\mathbf{x}\|=\|\mathbf{x}\|_2=\sqrt{\mathbf{x}\cdot\mathbf{x}}=\sqrt{x_1^2+x_2^2+\cdots+x_n^2}=\sqrt{\sum_{i=1}^n x_i^2}$$

- The length of a vector is defined to be its (Euclidean) norm.
- A unit vector is of length 1.
- Non-negativity properties also hold for the norm:

$$\forall\mathbf{x}\neq 0:\ \|\mathbf{x}\|^2>0 \qquad \|\mathbf{x}\|^2=0\Leftrightarrow\mathbf{x}=0$$

Bases and Transformations

- We will look at:
  - Linear Independence
  - Bases
  - Orthogonality
  - Change of basis (Linear Transformation)
  - Matrices and Matrix Operations

Linear Dependence

- Linear combination of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n$

$$c_1\mathbf{x}_1+c_2\mathbf{x}_2+\cdots+c_n\mathbf{x}_n$$

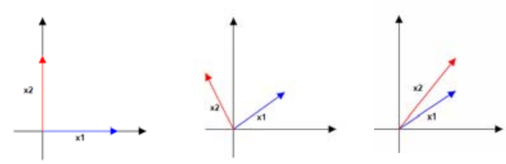
- A set of vectors $X=\{\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n\}$ are linearly dependent if there exists a vector $\mathbf{x}_i\in X$

that is a linear combination of the rest of the vectors.

Linear Dependence

- In $\mathbb{R}^n$
  - sets of n+1vectors are always dependent
  - there can be at most n linearly independent vectors

Bases (Examples in $\mathbb{R}^2$)



Bases

- A basis is a linearly independent set of vectors that spans the "whole space". ie., we can write every vector in our space as linear combination of vectors in that set.
- Every set of *n* linearly independent vectors in $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$
- A basis is called
  - ***orthogonal***, if every basis vector is orthogonal to all other basis vectors
  - ***orthonormal***, if additionally all basis vectors have length 1.

Bases

- Standard basis in $\mathbb{R}^n$ is made up of a set of unit vectors:

$$\hat{\mathbf{e}}_1 = \begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}, \hat{\mathbf{e}}_2 = \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}, \dots \hat{\mathbf{e}}_n = \begin{pmatrix}0\\0\\0\\\vdots\\1\end{pmatrix}.$$

- We can write a vector in terms of its standard basis:

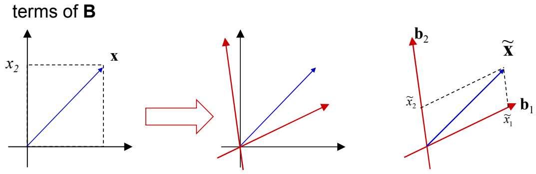
$$\begin{pmatrix}4\\7\\-3\end{pmatrix} = 4\,\hat{\mathbf{e}}_1 + 7\,\hat{\mathbf{e}}_2 - 3\,\hat{\mathbf{e}}_3$$

- Observation: -- to find the coefficient for a particular basis vector, we project our vector onto it.

$$x_i = \hat{\mathbf{e}}_i \cdot \mathbf{x}$$

Change of basis

- Suppose we have a new basis $\mathbf{B} = [\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_n]$, $\mathbf{b}_i \in \mathbb{R}^m$ and a vector $\mathbf{x} \in \mathbb{R}^m$ that we would like to represent in terms of **B**



- Compute the new components $\tilde{\mathbf{x}} = \mathbf{B}^{-1}\mathbf{x}$
- When B is orthonormal $\tilde{\mathbf{x}} = \begin{bmatrix}\mathbf{b}_1^T\mathbf{x}\\ \vdots \\ \mathbf{b}_n^T\mathbf{x}\end{bmatrix}$
  - $\tilde{\mathbf{x}}$ is a projection of **x** onto $\mathbf{b}_i$
  - Note the use of a dot product

Outer Product

$$\mathbf{x} \circ \mathbf{y} = \mathbf{x}\mathbf{y}^T = \begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix}\begin{bmatrix}y_1 & \cdots & y_m\end{bmatrix} = \mathbf{M}$$

- A matrix M that is the outer product of two vectors is a matrix of rank 1.

Matrix Multiplication – dot product

- Matrix multiplication can be expressed using dot products

$$\mathbf{BA} = \begin{bmatrix} \mathbf{b}_1^T \\ \vdots \\ \mathbf{b}_m^T \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \cdot \mathbf{a}_1 & & \mathbf{b}_1 \cdot \mathbf{a}_n \\ & \ddots & \\ \mathbf{b}_m \cdot \mathbf{a}_1 & & \mathbf{b}_m \cdot \mathbf{a}_n \end{bmatrix}$$

Matrix Multiplication – outer product

- Matrix multiplication can be expressed using a sum of outer products

$$\mathbf{BA} = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix} \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix}$$

$$= \mathbf{b}_1\mathbf{a}_1^T + \mathbf{b}_2\mathbf{a}_2^T + \cdots \mathbf{b}_n\mathbf{a}_n^T$$

$$= \sum_{i=1}^{n} \mathbf{b}_i \circ \mathbf{a}_i$$

Rank of a Matrix

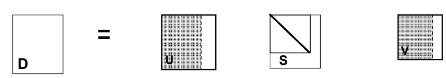
The rank of a matrix is the number of linearly independent rows or columns.

Examples: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has rank 2, but $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$ only has rank 1.

Equivalent to the dimension of the range of the linear transformation.

A matrix with full rank is called *non-singular*, otherwise it is singular.

Singular Value Decomposition: D=USV^T



- A matrix $\mathbf{D} \in \mathbb{R}^{I_1 \times I_2}$ has a column space and a row space
- SVD orthogonalizes these spaces and decomposes **D**

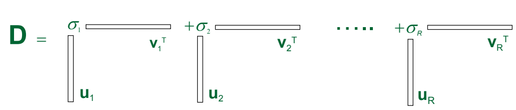
$\mathbf{D} = \mathbf{USV}^T$ ($\mathbf{U}$ contains the left singular vectors/eigenvectors)
($\mathbf{V}$ contains the right singular vectors/eigenvectors)

- Rewrite as a sum of a minimum number of rank-1 matrices

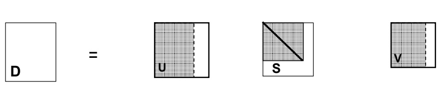
$$\mathbf{D} = \sum_{r=1}^{R} \sigma_r \mathbf{u}_r \circ \mathbf{v}_r$$

Matrix SVD Properties: D=USV^T

- Rank Decomposition: $\mathbf{D} = \sum_{r=1}^{R} \sigma_r \mathbf{u}_r \circ \mathbf{v}_r$
  - sum of min. number of rank-1 matrices



- Multilinear Rank Decomposition: $\mathbf{D} = \sum_{r_1=1}^{R_1} \sum_{r_2=1}^{R_2} \sigma_{r_1 r_2} \mathbf{u}_{r_1} \circ \mathbf{v}_{r_2}$



Matrix Inverse

A linear transformation can only have an inverse, if the associated matrix is non-singular.

The inverse $\mathbf{A}^{-1}$ of a matrix $\mathbf{A}$ is defined as:

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \quad (= \mathbf{A}\mathbf{A}^{-1})$$

We cannot cover here, how the inverse is computed. Nevertheless, it is similar to solving ordinary linear equation systems.

Some matrix properties

Matrix multiplication $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

For orthonormal matrices it holds that $\mathbf{A}^{-1} = \mathbf{A}^{\intercal}$.

For a diagonal matrix $\mathbf{D} = \mathrm{diag}\{d_1, \ldots, d_n\}$:

$$\mathbf{D}^{-1} = \mathrm{diag}\{d_1^{-1}, \ldots, d_n^{-1}\}$$

Matlab Tutorial