Math 6397 – Fall 2009 Name:

HOMEWORK #1

(1) Expose the fallacy of the following argument. Let $f \in L^2(\mathbb{R})$. By the properties of the Haar wavelet expansion we have that

$$f(x) = \sum_{j,k \in \mathbb{Z}} \langle f, H_{j,k} \rangle \, H_{j,k}(x).$$

Integrating on both sides

$$\begin{aligned} \int_{\mathbb{R}} f(x)\,dx &= \int_{\mathbb{R}} \sum_{j,k \in \mathbb{Z}} \langle f, H_{j,k} \rangle \, H_{j,k}(x)\,dx \\ &= \sum_{j,k \in \mathbb{Z}} \langle f, H_{j,k} \rangle \int_{\mathbb{R}} H_{j,k}(x)\,dx = 0. \end{aligned}$$

Hence every $f \in L^2(\mathbb{R})$ should satisfy $\int_{\mathbb{R}} f(x)\,dx = 0$. (Hint: The fallacy has nothing to do with f. We might even assume that $f \in L^1$ and the same observation would hold.)

(2) Let $\{\phi_k : k \in \mathbb{Z}\}$ be an ON system for $L^2(\mathbb{R})$. Prove that $\{\phi_k : k \in \mathbb{Z}\}$ is an ONB for $L^2(\mathbb{R})$ if and only if $f \in \overline{\mathrm{span}}\{\phi_k : k \in \mathbb{Z}\}$

3) Suppose that the function f on $\mathbb{R}$ is Hölder continuous with exponent α, $0 < \alpha \le 1$ on $\mathbb{R}$; that is, there is a constant $C > 0$ such that

$$|f(x) - f(y)| \le C\,|x - y|^{\alpha}$$

for all $x, y \in \mathbb{R}$. Show that $\langle f, H_{j,k} \rangle$ exists for all $j, k \in \mathbb{Z}$ and satisfies

$$|\langle f, H_{j,k} \rangle| \le C\,2^{-j(\alpha+1/2)}.$$