$\boxed{HW1}$

Let (φ_k) be an ON system in $L^2(\mathbb{R})$

(φ_k) is an ONB of $L^2(\mathbb{R})$ iff, for any $f \in L^2(\mathbb{R})$,

$$f = \sum \langle f, \varphi_k \rangle \varphi_k \qquad L^2\text{- convergence}$$

$\boxed{\text{Ex 2}}$ An ON system (φ_k) is an ONB of $L^2(\mathbb{R})$ iff, for every $f \in L^2(\mathbb{R})$, $f \in \overline{\operatorname{span}}\,(\varphi_k)$.

($\Leftarrow$) let $f \in \overline{\operatorname{span}}\,(\varphi_k)$

Given $\varepsilon > 0$, $\exists\ (c_1, \dots, c_N)$ s.t. $\quad \| f - \sum_{k=1}^{N} c_k \varphi_k \|^2 < \varepsilon$

For any finite sequence $(c_1, \dots, c_N)$

$$\| f - \sum_{k=1}^{N} \langle f, \varphi_k \rangle \varphi_k \|^2 \leq \| f - \sum_{k=1}^{N} c_k \varphi_k \|^2 + - \sum_{k=1}^{N} | c_k - \langle f, \varphi_k \rangle |^2 < \varepsilon$$

[This is a rather known ~~elementary~~ fact in Hilbert space theory.
A direct computation gives that, for any finite sequence (c_k),

$$\| f - \sum_{k=1}^{N} c_k \varphi_k \|^2 = \| f - \sum_{k=1}^{N} \langle f, \varphi_k \rangle \varphi_k \|^2 + \sum_{k=1}^{N} | c_k - \langle f, \varphi_k \rangle |^2 \ ]$$

Since $\quad \| f - \sum_{k=1}^{M} \langle f, \varphi_k \rangle \varphi_k \|^2 = \| f \|^2 - \sum_{k=1}^{M} | \langle f, \varphi_k \rangle |^2$

then $\quad \left(\| f - \sum_{k=1}^{M} \langle f, \varphi_k \rangle \varphi_k \| \right)$ is decreasing

and $\quad \| f - \sum_{k=1}^{M} \langle f, \varphi_k \rangle \varphi_k \| < \varepsilon \quad \forall M > N \quad$ and $\quad \sum_{k=1}^{N} \langle f, \varphi_k \rangle \varphi_k \to f$ in L^2 norm

($\Rightarrow$)

(φ_k) is ONB $\Rightarrow \quad \| f - \sum_{k=1}^{N} \langle f, \varphi_k \rangle \varphi_k \| \to 0 \quad$ as $N \to \infty$

Hence, given $\varepsilon > 0$, $\exists\ N(\varepsilon)$ s.t.

$$\| f - \sum_{k=1}^{N} \langle f, \varphi_k \rangle \varphi_k \| < \varepsilon$$

$\boxed{\text{Ex 3}}$

Suppose $|f(x) - f(y)| \leq C|x-y|^\alpha$, $\alpha \in (0,1]$ $\forall x, y \in \mathbb{R}$.

Since $\int H_{j,k}(t)\,dt = 0$, $\forall j,k \in \mathbb{Z}$, then

$$\langle f, H_{j,k} \rangle = \int f(t)\, H_{j,k}(t)\,dt = \int (f(t) - f(0))\, H_{j,k}(t)\,dt$$

Hence

$$|\langle f, H_{j,k} \rangle| \leq \int |f(t) - f(0)|\,|H_{j,k}(t)|\,dt$$

$$\leq C \int |t|^\alpha\, |H_{j,k}(t)|\,dt$$

$$= C \int_{2^{-j}k}^{2^{-j}(k+1)} |t|^\alpha\, 2^{j/2}\,dt$$

$$\leq C\, 2^{j/2} \left. \frac{t^{\alpha+1}}{\alpha+1} \right|_{2^{-j}k}^{2^{-j}(k+1)} = \frac{C}{\alpha+1}\, 2^{-j(\alpha+1)} \left((k+1)^{\alpha+1} - k^{\alpha+1}\right) 2^{j/2}$$

$$\leq C\, 2^{-j(\alpha + 1/2)}$$