SOLUTION HW 2

Ex 1 By the scaling equation,

$$|\hat{\varphi}(2\xi)|^2 + |\hat{\psi}(2\xi)|^2 = |\hat{\varphi}(\xi)|^2 \left(|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2\right) = |\hat{\varphi}(\xi)|^2$$

By iteration, $|\hat{\varphi}(\xi)|^2 = |\varphi(2^N\xi)|^2 + \sum_{j=1}^{N} |\hat{\psi}(2^j\xi)|^2$

Since $|\hat{\varphi}(\xi)| \le 1$, then $\left(\sum_{j=1}^{N} |\hat{\psi}(2^j\xi)|^2\right) = (a_N)$ is an INCREASING

and bounded sequence.

Hence (a_n) must converge.

$$\lim_N \int |\hat{\varphi}(2^N\xi)|^2 \, d\xi = \lim_N \frac{1}{2^N} \int |\hat{\varphi}(\xi)|^2 dx \to 0$$

By Fatou's lemma,

$$\int \lim_N |\hat{\varphi}(2^N\xi)|^2 d\xi \le \lim_N \int |\hat{\varphi}(2^N\xi)|^2 d\xi = 0$$

Hence $$|\hat{\varphi}(\xi)|^2 = \sum_{k=1}^{\infty} |\hat{\psi}(2^j\xi)|^2$$

and $$1 = \sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi+2\pi k)|^2 = \sum_{k,j\in\mathbb{Z}} |\hat{\psi}(2^j(\xi+2\pi k)|^2$$

Ex 2 Let $\hat{f}(\xi) = e^{i\xi} s(2\xi)\, \overline{m_0(\xi+\pi)}\, \hat{\varphi}(\xi) \in S$, $\hat{g}(\xi) = m(2\xi)\, m_0(\xi)\, \hat{\varphi}(\xi)$

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle = \int_{\mathbb{R}} e^{i\xi} s(2\xi)\, \overline{m_0(\xi+\pi)}\, \hat{\varphi}(\xi)\, \overline{m(2\xi)}\, \overline{m_0(\xi)}\, \overline{\hat{\varphi}(\xi)} \, d\xi$$

$$= \int_{\mathbb{R}} e^{i\xi} s(2\xi)\, \overline{m_0(\xi+\pi)}\, \overline{m_0(\xi)}\, \overline{m(2\xi)}\, |\hat{\varphi}(\xi)|^2 d\xi$$

$$= \underline{\sum_k} \int_0^{2\pi} e^{i\xi} s(2\xi)\, \overline{m_0(\xi+\pi)}\, \overline{m_0(\xi)}\, \overline{m(2\xi)}\, \underline{|\hat{\varphi}(\xi+2k\pi)|^2}\, d\xi$$

$\sum_k |\hat{\varphi}(\xi+2k\pi)|^2 = 1$ a.e.

$$= \int_0^{\pi} e^{i\xi} s(2\xi)\, \overline{m_0(\xi+\pi)}\, \overline{m_0(\xi)}\, \overline{m(2\xi)}\, d\xi + \int_{\pi}^{2\pi} e^{i\xi} s(2\xi)\, \overline{m_0(\xi+\pi)}\, \overline{m_0(\xi)}\, \overline{m(2\xi)}\, d\xi$$

$u = \xi - \pi$

$e^{-i\pi} = -1$

$$= \int_0^{\pi} s(2\xi) e^{i\xi}\, \overline{m(2\xi)} \left[\overline{m_0(\xi+\pi)}\, \overline{m_0(\xi)} - \overline{m_0(\xi)}\, \overline{m_0(\xi+\pi)}\right] d\xi = 0$$

Ex 3

Since $\varphi(x) \in L^1$ and has compact support, then $\sum_n \varphi(x+n)$ is locally L^1. Indeed, on each interval, there are only finitely many overlaps. Here $\sum_n \varphi(x+n)$ is $L^1([0,1])$ and periodic.

Let $\sigma(x) = \sum_{n\in\mathbb{Z}} \varphi(x+n) \in L^1([0,1])$ and compute its F-coefficients:

$$\int_0^1 \sigma(x)\, e^{-2\pi i k x}\, dx =$$

$$= \int_0^1 \sum_n \varphi(x+n)\, e^{-2\pi i k x}\, dx$$

$$= \sum_n \int_0^1 \varphi(x+n)\, e^{-2\pi i k x}\, dx \qquad y = x+n$$

$$= \sum_n \int_n^{n+1} \varphi(y)\, e^{-2\pi i k (y-n)}\, dy \qquad \left(e^{+2\pi i k n} = 1 \quad \forall n\right)$$

$$= \int_{\mathbb{R}} \varphi(y)\, e^{-2\pi i k y}\, dy = \hat{\varphi}(2\pi k)$$

Since φ is the scaling function of an MRA, then $\hat{\varphi}(2\pi k) = \begin{cases} 1 & k=0 \\ 0 & k\neq 0 \end{cases}$

All F-coefficients are 0 except for the coeff $k=0$, and, thus, $\sigma(x)$ is constant.

Since $\sigma(x)$ has period 1 and its 0-Fourier coefficient is 1, then $\sigma(x)=1$ $x\in[0,1]$