Math 3321 – Spring 2024 Name:

Homework #1

You must justify all steps to get credit for your work

Please submit the HW via CASA or email your completed assignment as a single PDF file to `jshi24@CougarNet.UH.EDU`.

(1)[3Pts] Consider the differential equation

$$y'' - 16y = 0$$

Is any of the following functions a solution for the differential equation?
(a). $y = \sin 4x$, (b). $y = \sinh 4x$, (c). $y = e^{-4x}$

Solution. (a).

1. From the expression of $y = \sin 4x$, we compute
$$y' = 4\cos 4x,\ y'' = -16\sin 4x.$$
2. Substitute into the left-hand side of ODE:
$$-16\sin 4x - 16\sin 4x = -32\sin 4x.$$
3. Since the result is DIFFERENT from the right-hand side of ODE, then $y = \sin 4x$ is NOT a solution.

(b).

1. From the expression of $y = \sinh 4x$, we compute
$$y' = 4\cosh 4x,\ y'' = 16\sinh 4x.$$
2. Substitute into the left-hand side of ODE:
$$16\sinh 4x - 16\sinh 4x = 0.$$
3. Since the result matches the right-hand side of ODE, then $y = \sinh 4x$ is a solution.

(c).

1. From the expression of $y = e^{-4x}$, we compute
$$y' = -4e^{-4x},\ y'' = 16e^{-4x}.$$
2. Substitute into the left-hand side of ODE:
$$16e^{-4x} - 16e^{-4x} = 0.$$
3. Since the result matches the right-hand side of ODE, then $y = e^{-4x}$ is a solution.

□

(2)[3Pts] Find the real numbers r such that $y = e^{rx}$ is a solution of

$$y'' - 6y' + 9y = 0$$

Solution. 1. From the expression of $y = e^{rx}$, we compute

$$y' = re^{rx},\ y'' = r^2 e^{rx}.$$

2. Substitute into the left-hand side of ODE:

$$r^2 e^{rx} - 6re^{rx} + 9e^{rx} = 0.$$

3. For the equation to be 0 for all x, it must be $r^2 - 6r - 9 = 0$. Hence it must be $r = 3$.

□

(3)[3Pts] Verify that $y = c_1 e^{2x} + c_2 e^{-3x}$ is the general solution of $y'' + y' - 6y = 0$ and find the solution of the IVP with initial conditions $y(0) = 1$, $y'(0) = 1$.

Solution. $y'' + y' - 6y = 0$ is a second-order homogeneous linear ODE with constant coefficients. We write the characteristic equation

$$r^2 + r - 6 = 0.$$

The roots are $r_1 = -3$ and $r_2 = 2$. We are in the situation of 2 distinct real roots. Thus the general solution is

$$y(x) = c_1 e^{2x} + c_2 e^{-3x}.$$

From the expression of $y(x) = c_1 e^{2x} + c_2 e^{-3x}$, we compute

$$y'(x) = 2c_1 e^{2x} - 3c_2 e^{-3x}.$$

By $y(0) = 1$ and $y'(0) = 1$, we have

$$\begin{cases} c_1 + c_2 = 1 \\ 2c_1 - 3c_2 = 1 \end{cases} \Rightarrow \begin{cases} c_1 = \frac{4}{5} \\ c_2 = \frac{1}{5} \end{cases}.$$

□

(4)[3Pts] Find the general solution of

$$y' - 4y = e^{-x}$$

Solution. 1. We identify

$$p(x) = -4,\ q(x) = e^{-x}.$$

2. We compute the integrating factor

$$h(x) = \int p(x)dx = -4x.$$

3. We write the general solution

$$\begin{aligned} y(x) &= e^{-h(x)} \int e^{h(x)} q(x) dx + C e^{-h(x)} \\ &= e^{4x} \int e^{-5x} dx + C e^{4x} \\ &= -\frac{1}{5} e^{-x} + C e^{4x}. \end{aligned}$$

□

(5)[4Pts] Find the general solution of

$$x^2 y' + 5xy = x^{-3} \cos(3x)$$

Solution. 1. We write the ODE as $y' + \frac{5}{x} y = \frac{\cos(3x)}{x^5}$ and identify

$$p(x) = \frac{5}{x},\ q(x) = \frac{\cos(3x)}{x^5}.$$

2. We compute the integrating factor

$$h(x) = \int p(x) dx = \int \frac{5}{x} dx = \ln x^5.$$

3. We write the general solution

$$\begin{aligned} y(x) &= e^{-h(x)} \int e^{h(x)} q(x) dx + C e^{-h(x)} \\ &= x^{-5} \int \cos 3x dx + C x^{-5} \\ &= \frac{1}{3} x^{-5} \sin 3x + C x^{-5}. \end{aligned}$$

□

(6)[4Pts] Find the general solution of

$$xy' - y = \frac{5}{2} x \ln x$$

Solution. 1. We write the ODE as $y' - \frac{1}{x} y = \frac{5}{2} \ln x$ and identify

$$p(x) = -\frac{1}{x},\ q(x) = \frac{5}{2} \ln x.$$

2. We compute the integrating factor

$$h(x) = \int p(x)dx = \int \frac{1}{x}dx = -\ln x.$$

3. We write the general solution

$$\begin{aligned} y(x) &= e^{-h(x)} \int e^{h(x)} q(x)dx + Ce^{-h(x)} \\ &= \frac{5}{2}x \int \frac{\ln x}{x}dx + Cx \\ &= \frac{5}{4}x \ln^2 x + Cx. \end{aligned}$$

□