Math 3321 – Spring 2024 Name:

Homework #2

You must justify all steps to get credit for your work

Please submit the HW via CASA or email your completed assignment as a single PDF file to `jshi24@CougarNet.UH.EDU`.

(1)[4Pts] Find the general solution and solve the following IVP

$$6y' - 2y = xy^4, \quad y(0) = -2$$

First get the differential equation in the proper form and then write down the substitution.

$$6y^{-4}y' - 2y^{-3} = x \quad \Rightarrow \quad v = y^{-3}, \quad v' = -3y^{-4}y'$$

Plugging the substitution into the differential equation gives,

$$-2v' - 2v = x \quad \Rightarrow \quad v' + v = -\frac{1}{2}x, \quad \mathbf{e}^{h(x)} = \mathbf{e}^{\int 1dx} = \mathbf{e}^{x}$$

Again, we've rearranged a little and given the integrating factor needed to solve the linear differential equation. Upon solving the linear differential equation we have,

$$v(x) = -\frac{1}{2}(x-1) + c\mathbf{e}^{-x}$$

Now back substitute to get back into y 's.

$$y^{-3} = -\frac{1}{2}(x-1) + c\mathbf{e}^{-x} \Rightarrow y(x) = -\left(\frac{2}{x-1+2c\mathbf{e}^{-x}}\right)^{\frac{1}{3}}$$

Now we need to apply the initial condition and solve for c.

$$-\frac{1}{8} = \frac{1}{2} + c \quad \Rightarrow \quad c = -\frac{5}{8}$$

Plugging in c and solving for y gives,

$$y(x) = -\frac{2}{(4x-4+5\mathbf{e}^{-x})^{\frac{1}{3}}}$$

(2)[4Pts] Find the general solution and solve the following IVP

$$y' + \frac{1}{x}y - \sqrt{y} = 0, \quad y(1) = 0$$

Let's first get the differential equation into proper form.

$$y' + \frac{1}{x}y = y^{\frac{1}{2}} \quad \Rightarrow \quad y^{-\frac{1}{2}}y' + \frac{1}{x}y^{\frac{1}{2}} = 1$$

The substitution is then,

$$v = y^{\frac{1}{2}} \quad v' = \frac{1}{2}y^{-\frac{1}{2}}y'$$

Now plug the substitution into the differential equation to get,

$$2v' + \frac{1}{x}v = 1 \quad \Rightarrow \quad v' + \frac{1}{2x}v = \frac{1}{2}, \quad \mathbf{e}^{h(x)} = \mathbf{e}^{\int \frac{1}{2x}dx} = x^{\frac{1}{2}}$$

As we've done with the previous examples we've done some rearranging and given the integrating factor needed for solving the linear differential equation. Solving this gives us,

$$v(x) = \frac{1}{3}x + cx^{-\frac{1}{2}}$$

In terms of y this is,

$$y^{\frac{1}{2}} = \frac{1}{3}x + cx^{-\frac{1}{2}}, \quad \Rightarrow \quad y(x) = \left(\frac{1}{3}x + cx^{-\frac{1}{2}}\right)^2$$

Applying the initial condition and solving for c gives,

$$0 = \frac{1}{3} + c \quad \Rightarrow \quad c = -\frac{1}{3}$$

Plugging in for c and solving for y gives us the solution.

$$y(x) = \left(\frac{1}{3}x - \frac{1}{3}x^{-\frac{1}{2}}\right)^2 = \frac{x^3 - 2x^{\frac{3}{2}} + 1}{9x}$$

(3)[4Pts] Find the general solution of the following problem

$$xyy' + 4x^2 + y^2 = 0$$

Let's first divide both sides by x^2 to rewrite the differential equation as follows,

$$\frac{y}{x}y' = -4 - \frac{y^2}{x^2} = -4 - \left(\frac{y}{x}\right)^2$$

Now, this is not in the officially proper form as we have listed above, but we can see that everywhere the variables are listed they show up as the ratio, y/x and so this is really as far as we need to go. So, let's plug the substitution into this form of the differential equation to get,

$$v\left(v + xv'\right) = -4 - v^2$$

Next, rewrite the differential equation to get everything separated out.

$$vxv' = -4 - 2v^2$$

$$xv' = -\frac{4 + 2v^2}{v}$$

$$\frac{v}{4 + 2v^2}dv = -\frac{1}{x}dx$$

Integrating both sides gives,

$$\frac{1}{4}\ln\left(4 + 2v^2\right) = -\ln(x) + c$$

We need to do a little rewriting using basic logarithm properties in order to be able to easily solve this for v.

$$\ln\left(4 + 2v^2\right)^{\frac{1}{4}} = \ln(x)^{-1} + c$$

Now exponentiate both sides and do a little rewriting

$$\left(4+2v^2\right)^{\frac{1}{4}} = \mathbf{e}^{\ln(x)^{-1}+c} = \mathbf{e}^c\mathbf{e}^{\ln(x)^{-1}} = \frac{c}{x}$$

Note that because c is an unknown constant then so is $\mathbf{e}^c$ and so we may as well just call this c as we did above. Finally, let's solve for v and then plug the substitution back in and we'll play a little fast and loose with constants again.

$$\begin{aligned} 4+2v^2 &= \frac{c^4}{x^4} = \frac{c}{x^4} \\ v^2 &= \frac{1}{2}\left(\frac{c}{x^4}-4\right) \\ \frac{y^2}{x^2} &= \frac{1}{2}\left(\frac{c-4x^4}{x^4}\right) \\ y^2 &= \frac{1}{2}x^2\left(\frac{c-4x^4}{x^4}\right) = \frac{c-4x^4}{2x^2} \end{aligned}$$

(4)[4Pts] Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that the temperature of a cup of tea obeys Newton's law of cooling. If the coffee has a temperature of 200F when freshly poured, and 1 min later has cooled to 190F in a room at 70F. Determine when the coffee reaches a temperature of 150F.

Let T be the temperature of the object, and T_s the temperature of the surroundings. We can write Newton's law of cooling in equation form as follows:

$$\frac{dT}{dt} = -k\left(T-T_s\right).$$

Here k is a still-unknown constant, greater than zero. Let's double-check the sign: if the object has a higher temperature than its surroundings, then $T > T_s$, so dT/dt is negative, so the object is cooling, which is what we expect. We solve the equation by separating the variables.

$$\begin{aligned} \int \frac{1}{T-T_s}dT &= \int -kdt \\ \ln|T-T_s| &= -kt + C \\ |T-T_s| &= Ae^{-kt} \\ T-T_s &= Ae^{-kt} \quad (\text{ replacing } A \text{ with } \pm A) \\ T &= Ae^{-kt}+T_s \end{aligned}$$

In the given problem (with units degrees Fahrenheit), we have $T_s = 70$ and $T(0) = 200$, so we get $A = 130$. Since $T(1) = 190$,

$$\begin{aligned} 190 &= 130e^{-k}+70 \\ 120 &= 130e^{-k} \\ 120/130 &= e^{-k} \\ k &= \ln(120/130) \approx 0.08 \end{aligned}$$

Finally, we solve for t in $T(t) = 150$.

$$\begin{aligned} 150 &= 130e^{-0.08t} + 70 \\ 80 &= 130e^{-0.08t} \\ 80/130 &= e^{-0.08t} \\ \ln(80/130) &= -0.08t \\ t &= 6.066 \text{ min.} \end{aligned}$$

(5)[4Pts] The size of a certain bacterial colony increases at a rate proportional to the size of the colony. Suppose the colony occupied an area of 0.25 square centimeters initially, and after 8 hours it occupied an area of 0.35 square centimeters.

(a) Estimate the size of the colony t hours after the initial measurement.

(b) What is the expected size of the colony after 12 hours?

(c) Find the doubling time of the colony

(a). Let $P(t)$ denote the size of the colony t hours after the initial measurement. Since $P(0) = 0.25$ and $P(8) = 0.35$, we have

$$P(t) = P(0)\mathbf{e}^{kt} = 0.25\mathbf{e}^{kt}$$

$$P(8) = 0.25\mathbf{e}^{8k} = 0.35$$

Thus

$$\mathbf{e}^{8t} = \frac{7}{5} \quad \Rightarrow \quad k = \frac{\ln(7/5)}{8} \approx 0.0421$$

and

$$P(t) = 0.25\mathbf{e}^{0.0421t} \textit{ or } 0.25 \times \left(\frac{7}{5}\right)^{\frac{t}{8}}$$

(b).

$$P(12) = 0.25\mathbf{e}^{0.0421\times 12} \approx 0.414 \ cm^2$$

(c). The doubling time is

$$T = \frac{\ln 2}{k} \approx \frac{\ln 2}{0.0421} \approx 16.464 \textit{ hours.}$$