Math 3321 – Spring 2024 Name: SOLUTION

Homework #4

You must justify all steps to get credit for your work

Please submit the HW via CASA or email your completed assignment as a single PDF file to `jshi24@CougarNet.UH.EDU`.

(1)[5Pts] Consider the following linear second-order differential equation

$$xy'' - (x+1)y' + y = 0, \qquad x > 0$$

(a) Show that $y_1 = e^x$ and $y_2 = x + 1$ are solutions of the differential equation above.
(b) Use the method of variation of parameters to find a particular solution of

$$xy'' - (x+1)y' + y = x^2 e^{2x}, \qquad x > 0$$

(c) Write the general solution of the non-homogeneous equation in part (b)

(a)

$$y_1 = e^x \Rightarrow y_1' = e^x,\ y_1'' = e^x \Rightarrow xy_1'' - (x+1)y_1' + y_1 = e^x(x - (x+1) + 1) = 0.$$

$$y_2 = x + 1 \Rightarrow y_2' = 1,\ y_2'' = 0 \Rightarrow xy_2'' - (x+1)y_2' + y_2 = 0 - (x+1) + (x+1) = 0.$$

(b) Let's first get the differential equation into proper form:

$$y'' - \frac{x+1}{x}y' + \frac{1}{x}y = xe^{2x}, \qquad x > 0.$$

By (a), we compute the Wronskian

$$W(x) = y_1 y_2' - y_2 y_1' = -xe^x.$$

By the method of variation of parameters, a particular solution is

$$z(x) = uy_1 + vy_2,$$

where

$$u' = \frac{-y_2 f}{W},\ v' = \frac{y_1 f}{W} \text{ and } f = xe^{2x}.$$

We compute

$$u(x) = \int \frac{-y_2 f}{W} dx = \int \frac{-(x+1)xe^{2x}}{-xe^x} dx = xe^x,$$

$$v(x) = \int \frac{y_1 f}{W} dx = \int \frac{e^x xe^{2x}}{-xe^x} dx = -\frac{1}{2}e^{2x}.$$

Hence

$$z(x) = uy_1 + vy_2 = \frac{1}{2}(x-1)e^{2x}.$$

(c)

$$y(x) = Ae^x + B(x+1) + \tfrac{1}{2}(x-1)e^{2x}.$$

(2)[4Pts] Find the general solution of the following differential equation

$$y'' - 3y' + 2y = x^2$$

We solve the homogeneous equation first. The characteristic equation is

$$p(r) = r^2 - 3r + 2 = 0, \quad r = 2, r = 1.$$

Hence the solution to the homogeneous problem is

$$y_h(x) = Ae^{2x} + Be^x.$$

Based on the form of the function on the right hand side, the particular solution should have the form

$$z = A_0 + A_1 x + A_2 x^2.$$

We have

$$z' = A_1 + 2A_2 x, \text{ and } z'' = 2A_2.$$

Substitution into the differential equation gives

$$2A_2 - 3(A_1 + 2A_2 x) + 2(A_0 + A_1 x + A_2 x^2) = x^2.$$

Hence we have

$$\begin{cases} A_0 = \frac{7}{4} \\ A_1 = \frac{3}{2} \\ A_2 = \frac{1}{2} \end{cases} .$$

It follows that

$$z = \tfrac{1}{4}(2x^2 + 6x + 7).$$

Thus the general solution is

$$y = Ae^{2x} + Be^x + \tfrac{1}{4}(2x^2 + 6x + 7).$$

(3)[4Pts] Find the general solution of the following differential equation

$$y'' - y' - 2y = e^{-x} + x^2 + \cos(x)$$

We solve the homogeneous equation first. The characteristic equation is

$$p(r) = r^2 - r - 2 = 0, \quad r = -1, 2.$$

Hence the solution to the homogeneous problem is

$$y_h = Ae^{-x} + Be^{2x}.$$

This implies that $z = Ce^{-x}$ is the solution of the reduced equation. Thus, to find a particular solution of

$$y'' - y' - 2y = e^{-x},$$

we need to choose a solution of the form

$$z_1 = Cxe^{-x}.$$

Repeating the process of Problem # 2, we solve

$$C = -\frac{1}{3} \Rightarrow z_1 = -\frac{1}{3}xe^{-x}.$$

To find a particular solution of

$$y'' - y' - 2y = x^2,$$

we need to choose a solution of the form

$$z_2 = A_0 + A_1x + A_2x^2.$$

Repeating the process of Problem # 2, we solve

$$\begin{cases} A_0 = -\frac{3}{4} \\ A_1 = \frac{1}{2} \\ A_2 = -\frac{1}{2} \end{cases} \quad \Rightarrow z_2 = \frac{1}{4}(-2x^2 + 2x - 3).$$

To find a particular solution of

$$y'' - y' - 2y = \cos(x),$$

we need to choose a solution of the form

$$z_3 = B_1 \cos(x) + B_2 \sin(x).$$

Repeating the process of Problem # 2, we solve

$$\begin{cases} B_1 = -\frac{3}{10} \\ B_2 = -\frac{1}{10} \end{cases} \quad \Rightarrow z_2 = -\frac{3}{10}\cos(x) - \frac{1}{10}\sin(x).$$

Hence, combining these observations, we have that the particular solution has the form

$$z = z_1 + z_2 + z_3 = -\frac{1}{3}xe^{-x} - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{3}{4} - \frac{3}{10}\cos(x) - \frac{1}{10}\sin(x).$$

Thus the general solution is

$$y = y_h + z = Ae^{-x} + Be^{2x} - \frac{1}{3}xe^{-x} - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{3}{4} - \frac{3}{10}\cos(x) - \frac{1}{10}\sin(x).$$

(4)[4Pts] Find the general solution of the following differential equation

$$y'' - 4y = 3e^{2x} + 4e^{-x}$$

We solve the homogeneous equation first. The characteristic equation is

$$p(r) = r^2 - 4 = 0, \quad r = \pm 2.$$

Hence the solution to the homogeneous problem is

$$y_h = Ae^{-2x} + Be^{2x}.$$

This implies that $z = Ce^{2x}$ *is the solution of the reduced equation. Thus, to find a particular solution of*

$$y'' - 4y = 3e^{2x},$$

we need to choose a solution of the form

$$z_1 = Cxe^{2x}.$$

Repeating the process of Problem # 2, we solve

$$C = \tfrac{3}{4} \Rightarrow z_1 = \tfrac{3}{4}xe^{2x}.$$

To find a particular solution of

$$y'' - 4y = 4e^{-x},$$

we need to choose a solution of the form

$$z_2 = De^{-x}.$$

Repeating the process of Problem # 2, we solve

$$D = -\tfrac{4}{3} \Rightarrow z_2 = -\tfrac{4}{3}e^{-x}.$$

Hence, combining these observations, we have that the particular solution has the form

$$z = z_1 + z_2 = \tfrac{3}{4}xe^{2x} - \tfrac{4}{3}e^{-x}.$$

Thus the general solution is

$$y = y_h + z = Ae^{-2x} + Be^{2x} + \tfrac{3}{4}xe^{2x} - \tfrac{4}{3}e^{-x}.$$

(5)[3Pts] Give the form of a particular solution for

$$y'' - 4y' + 5y = 1 + x^2 + e^{2x}\cos x$$

We solve the homogeneous equation first. The characteristic equation is

$$p(r) = r^2 - 4r + 5 = 0, \quad r = 2 \pm i.$$

Hence the solution to the homogeneous problem is

$$y_h = e^{2x}(A\cos(x) + B\sin(x)).$$

This implies that $z = e^{2x}(A\cos(x) + B\sin(x))$ *is the solution of the reduced equation. Thus, to find a particular solution of*

$$y'' - 4y' + 5y = e^{2x}\cos x,$$

we need to choose a solution of the form

$$z_1 = xe^{2x}(A\cos(x) + B\sin(x)).$$

To find a particular solution of

$$y'' - 4y' + 5y = 1 + x^2,$$

we need to choose a solution of the form

$$z_2 = C + Dx + Ex^2.$$

Hence, combining these observations, we have that the particular solution has the form

$$z = z_1 + z_2 = xe^{2x}(A\cos(x) + B\sin(x)) + (C + Dx + Ex^2).$$