Math 3321 – Spring 2024 Name: SOLUTION

Homework #5

You must justify all steps to get credit for your work

Please submit the HW via CASA or email your completed assignment as a single PDF file to `jshi24@CougarNet.UH.EDU`.

(1)[4Pts] Consider the following linear differential equation

$$y''' + y = 0$$

(a) Find the general solution of the following differential equation
(b) Solve the IVP where $y(0) = 0, y'(0) = 1, y''(0) = 0$

(a) Characteristic polynomial

$$p(r) = r^3 + 1$$

whose 3 roots are

$$r = \sqrt[3]{-1} = e^{i\pi/3}, e^{i\pi}, e^{i5\pi/3}$$

We can re-write the roots as $r_1 = -1$ *and* $r_{2,3} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. *Thus*

$$y(x) = C_1 e^{-x} + C_2 e^{x/2}\cos(\tfrac{\sqrt{3}}{2}x) + C_3 e^{x/2}\sin(\tfrac{\sqrt{3}}{2}x)$$

(b)

$$y(x) = -\tfrac{1}{3}e^{-x} + \tfrac{1}{3}e^{x/2}\cos(\tfrac{\sqrt{3}}{2}x) + \tfrac{\sqrt{3}}{3}e^{x/2}\sin(\tfrac{\sqrt{3}}{2}x)$$

(2) Use the method of variation of parameters to compute a particular solution of the following linear differential equation

$$y''' - y' = x$$

Characteristic polynomial

$$p(r) = r^3 - r$$

whose 3 roots are

$$r = 0, 1, -1$$

The non-homogeneous term is a polynomial of degree 1. Since A *is a solution of the homogeneous equation, we look for a particular solution of the form* $y_p = x(A + Bx) = Ax + Bx^2$. *We find* $B = 1/2$, *hence*

$$y_p(x) = \tfrac{1}{2}x^2$$

(3) Use the method of variation of parameters to compute a particular solution of the following linear differential equation (note that $r = 1$ *is a root of the characteristic polynomial)*

$$y^{(4)} - 4y^{(3)} + 6y'' - 4y' + y = e^x$$

Characteristic polynomial

$$p(r) = r^4 - 4r^3 + 6r^2 - 4r + 1 = (r-1)^4$$

whose root $r = 1$ has multiplicity 4, so the homogeneous solution is

$$y_h(x) = (A + Bx + Cx^2 + Dx^3)e^x$$

The non-homogeneous term e^x is contained in the homogeneous solution so we look for a particular solution of the form $y_p = Ex^4e^x$. By substitution, we find $E = 1/24$, hence

$$y_p(x) = \tfrac{1}{24}x^4e^x$$

(4) Find the general solution of the following non-homogeneous differential equation

$$y''' + y'' + y' = x$$

Characteristic polynomial

$$p(r) = r^3 + r^2 + r = r(r^2 + r + 1)$$

whose roots are

$$r_1 = 0, r_{2,3} = -\tfrac{1}{2} \pm i\tfrac{\sqrt{3}}{2}$$

Hence the homogeneous solution is

$$y_h(x) = C_1 + C_2e^{-x/2}\cos(\tfrac{\sqrt{3}}{2}x) + C_3e^{-x/2}\sin(\tfrac{\sqrt{3}}{2}x)$$

The non-homogeneous term x is not contained in the homogeneous solution but $f(x) = x = xe^{0x}$ and $r_1 = 0$ is the root of characteristic equation $p(r) = 0$ so we look for a particular solution of the form $y_p = x(A + Bx)$. By substitution, we find $A = 1/2$ and $B = -1$, hence the general solution is

$$y(x) = C_1 + C_2e^{-x/2}\cos(\tfrac{\sqrt{3}}{2}x) + C_3e^{-x/2}\sin(\tfrac{\sqrt{3}}{2}x) + \left(\tfrac{1}{2}x^2 - x\right)$$

(5) Find the solution of the following IVP modeling undamped forced harmonic motion

$$y'' + 4y = \sin(2x), \quad y(0) = 3/4, \quad y'(0) = 2$$

Please, write the solution using sinusoidal functions with amplitude and phase as in Lecture 12.

Characteristic polynomial

$$p(r) = r^2 + 4$$

whose roots are

$$r = \pm 2i$$

Hence the homogeneous solution is

$$y_h(x) = C_1\cos(2x) + C_2\sin(2x) = A\sin(2x + \phi)$$

The forcing term is contained in the homogeneous solution so we look for a particular solution of the form

$$y_p(x) = Bx\sin(2x + \psi)$$

By substitution, we find $B = -1/4$, $\psi = \pi/2$ *hence the general solution is*

$$y(x) = C_1\cos(2x) + C_2\sin(2x) - \tfrac{1}{4}x\cos(2x)$$

We have

$$\tfrac{3}{4} = y(0) = C_1, \quad 2 = y'(0) = 2C_2 - \tfrac{1}{4}$$

This implies

$$\begin{cases} C_1 = \frac{3}{4} \\ C_2 = \frac{9}{8} \end{cases}$$

and

$$y(x) = \tfrac{3}{4}\cos(2x) + \tfrac{9}{8}\sin(2x) - \tfrac{1}{4}x\cos(2x)$$