Math 3321 – Spring 2024 Name: SOLUTION

Test #1

You must show your work and report all steps of your solution to get credit for your work

(1)[4Pts] Determine the values of the constant $a \in \mathbb{R}$, if any, such that $y(x) = xe^{ax}$ is a solution of the following differential equation

$$xy'' - xy' - y = 0$$

By direct calculation, we have that

$$y'(x) = (1 + ax)e^{ax}, \quad y''(x) = (2a + a^2x)e^{ax}$$

Substituting into the differential equation, we obtain

$$x(2a + a^2x)e^{ax} - x(1 + ax)e^{ax} - xe^{ax} = 0$$

This implies

$$(a^a - a)x^2 + (2a - 2)x = 0$$

Since both the coefficients of x and x^2 must vanish, we conclude $a = 1$. Thus we conclude that $y(x) = xe^x$ is a solution.

(2)[4Pts] Find the general solution of the following differential equation

$$y' + x \tan y = 0$$

It is a differential equation with separable variables. We can write it as

$$y' = -x \tan y$$

Hence, under the assumption that $y \neq 0$ ($y = 0$ is a solution, as you can see by substitution), we have

$$\frac{dy}{\tan y} = -xdx \Rightarrow \int \frac{\cos x}{\sin x} dy = -\int x\, dx$$

We obtain

$$\ln|\sin y| = -\tfrac{1}{2}x^2 + C$$

Hence

$$|\sin y| = e^C\, e^{-x^2/2}$$

Hence

$$\sin y = \pm e^C\, e^{-x^2/2} = Ke^{-x^2/2},$$

where K is a non-zero constant. Since $y = 0$ is also a solution, we can allow $K \in \mathbb{R}$. Finally, we can write explicitly the general solution as

$$y = \arcsin(Ke^{-x^2/2})$$

with $K \in \mathbb{R}$.

(3)[5Pts] (a) Find the general solution of the following differential equation

$$y' = y \sin x + \sin 2x;$$

(b) next find a particular solution satisfying the initial condition $y(0) = -2$. *Write as*

$$y' - y \sin x - \sin 2x = 0$$

It is a linear first order differential equation. We solve it with the method of the integrating factor

$$u(x) = e^{-\int \sin x \, dx} = e^{\cos x}$$

Hence the general solution is

$$\begin{aligned}
y(x) &= e^{-\cos x} \int e^{\cos x} \sin(2x) \, dx \\
&= e^{-\cos x} 2 \int e^{\cos x} \cos x \sin x \, dx \\
&\quad (t = \cos x \Rightarrow -\sin x \, dx = dt) \\
&= -2e^{-\cos x} \int te^t \, dt \\
&= -2e^{-\cos x}(te^t - e^t + C) \\
&= -2e^{-\cos x}(\cos x \, e^{\cos x} - e^{\cos x} + C) \\
&= -2\cos x + 2 - 2Ce^{-\cos x}
\end{aligned}$$

Imposing the condition $y(0) = -2$, we obtain $C = e$. Thus the IVP solution is

$$y(x) = -2\cos x + 2 - 2e^{1-\cos x}$$

(4)[4Pts] Find the general solution of the following differential equation

$$3xy' + 9y = 2xy^{5/3}$$

Write the equation as

$$y' + \frac{3}{x} y = \frac{2}{3} y^{5/3}$$

This is a Bernoulli equation with $r = 5/3$. Setting $v = y^{1-r} = y^{-2/3}$, we obtain the linear ODE

$$v' + (1-r)\frac{3}{x} v = (1-r)\frac{2}{3} \quad \Rightarrow \quad v' - \frac{2}{x} v = -\frac{4}{9}$$

We apply the method of integrating factor with

$$u(x) = e^{-\int \frac{2}{x} dx} = x^{-2}$$

Hence

$$\begin{aligned}
v(x) &= x^2 \int (-\frac{4}{9}) x^{-2} dx + Cx^2 \\
= &= x^2 (\frac{4}{9}) x^{-1} + Cx^2 \\
&= \frac{4}{9} x + Cx^2.
\end{aligned}$$

Since $y = v^{-3/2}$, then the general solution is

$$y(x) = \left(\frac{4}{9}x + Cx^2\right)^{-3/2} = \left(\frac{9}{4x + C'x^2}\right)^{3/2}$$

where $C' \in \mathbb{R}$.

(5)[4Pts+2extra credit points for part b] (a) Find the general solution of the following differential equation

$$(x^2 + y^2)y' = 2xy,$$

[Hint: it can be written as a homogeneous equation and solved with the substitution $v = y/x$] and (b) solve the IVP determined by the initial condition $y(1) = -1$.

Part (a)

We write the equation as

$$y' = \frac{2xy}{x^2 + y^2} = f(x, y)$$

For any $\lambda > 0$, we have that $f(\lambda x, \lambda y) = f(x, y)$. Thus, we can use the substitution $v = y/x$, obtaining the differential equation

$$v' = \frac{f(1, v) - v}{x} = \frac{\frac{2v}{1+v^2} - v}{x} = \frac{v - v^3}{(1 + v^2)x}$$

Under the assumption $v \neq 0, v \neq 1$, this gives the separable differential equation

$$\frac{1 + v^2}{v - v^3} dv = \frac{1}{x} dx \quad \Rightarrow \quad \int \frac{1 + v^2}{v - v^3} dv = \int \frac{1}{x} dx \quad \int (\frac{1}{v} + \frac{2v}{1 - v^2}) dv = \int \frac{1}{x} dx$$

Hence, integrating both side we obtain

$$\ln |\frac{v}{v^2 - 1}| = \ln |x| + C$$

which gives

$$\frac{v}{v^2 - 1} = Kx$$

Solving for v, we obtain the solution

$$v = \frac{1 \pm \sqrt{1 + 4K^2x^2}}{2Kx}$$

Finally, with the substitution $y = vx$, we have the general solution

$$y(x) = \frac{1 \pm \sqrt{1 + 4K^2x^2}}{2K}$$

valid for any $K \neq 0$. We still need to check the cases $v = 0, v = 1$. Direct substitution shows that $y = 0$ and $y = \pm x$ are also solution of the differential equation. None of these is included in the general solution so they are all singular solutions.

Part (b)

The IVP with initial condition $y(x) = -1$ is satisfied by the singular solution $y = -x$.

Direct calculation shows that the is no value of K such that the general solution satisfies the initial condition.

(6)[4Pts] Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that the temperature of an iron rod removed from an oven obeys Newton's law of cooling. If the iron rod has a temperature of 200 C when removed from the oven and 1 min later has cooled to 170 C in a room at 25 C, determine the time in minutes when the iron rod reaches a temperature of 100 C. Please, round your solution to 2 decimal digits.

Let T be the temperature of the rod and T_s be the temperature of the surroundings. The Newton's law of cooling satisfies

$$\frac{dT}{dt} = -k(T - T_s)$$

so that we have the equation

$$T(t) = Ae^{-kt} + T_s$$

In our problem, $T_s = 25$ and $T(0) = 200$, so that $A = T(0) - T_s = 175$. Hence, we can compute the value of k using the observation

$$170 = 175e^{-k} + 25 \quad \Rightarrow \quad e^{-k} = 145/175 \quad \Rightarrow \quad k = \ln(175/145) \approx 0.188$$

Now we can look for the value of the time t such that $T(t) = 100$. We set the equation

$$100 = 175e^{-kt} + 25 \quad \Rightarrow \quad e^{-kt} = 75/175 \quad \Rightarrow \quad t = \frac{1}{k}\ln(175/75) \approx 4.51[\text{min}]$$