Math 3321 – Spring 2024 Name: SOLUTION

Test #2

You must show your work and report all steps of your solution to get credit for your work

(1)[4Pts] Consider the following differential equation

$$y'' - 4y' + 5y = 0$$

(a) Find the general solution.
(b) Solve the IVP with $y(0) = 0$ and $y'(0) = -1$.

(a) Characteristic polynomial:

$$p(r) = r^2 - 4r + 5$$

Roots: $r_{1,2} = 2 \pm i$
General solution:

$$y(x) = e^{2x}(A\cos x + B\sin x)$$

(b)

$$0 = y(0) = A \quad \Rightarrow A = 0$$

$$-1 = y'(0) = (2B - A)e^{2x}\sin x + (2A + B)e^{2x}\cos x)|_{x=0} = 2A + B \quad \stackrel{A=0}{\Rightarrow} B = -1$$

Hence, the IVP solution is

$$y(x) = -e^{2x}\sin x$$

(2)[4Pts] Find the general solution of the following differential equation

$$y'' + 4y = \sin(2x)$$

We first find the solution of the homogeneous problem. The characteristic polynomial is

$$r^2 + 4 = 0$$

Roots are $r = \pm 2i$, *hence the solution of the homogeneous problem is*

$$y_h(x) = A\cos(2x) + B\sin(2x)$$

Since $\sin(2x)$ *is already a solution of the homogeneous problem, we look for a particular solution of the form*

$$y_p(x) = x(C_1\cos(2x) + C_2\sin(2x))$$

The method of substitution yields: $C_1 = -1/4$, $C_2 = 0$. *Hence* $y_p(x) = -\frac{1}{4}x\cos(2x)$ *and, thus, the general solution is*

$$y(x) = A\cos(2x) + B\sin(2x) - \frac{1}{4}x\cos(2x)$$

(3)[3Pts] Without computing the constants, give the form of a particular solution for

$$y'' - 4y' + 5y = x - 7e^{2x} + e^{2x}\sin x$$

From the solution of problem 1, we have that the solution of the homogeneous problem is

$$y(x) = e^{2x}(A\cos x + B\sin x)$$

The forcing term is the sum of 3 functions:

$$y_1(x) = x, \quad y_2(x) = -7e^{2x}, \quad y_3(x) = e^{2x}\sin x$$

y_1 *and* y_2 *are not contained in the solution of the homogeneous problem. Hence, the form of particular solutions corresponding to these terms are*

$$y_{p1}(x) = A + Bx, \quad y_{p2}(x) = Ce^{2x}$$

Since $e^{2x}\sin x$ *is already a solution of the homogeneous problem, here we look for a particular solution of the form*

$$y_{p3}(x) = xe^{2x}(D\cos(x) + E\sin(x))$$

Hence, combining the 3 terms, a particular solution is of the form

$$y_p = A + Bx + Ce^{2x} + xe^{2x}(D\cos(x) + E\sin(x))$$

(4)[4Pts] Find the general solution of the following non-homogeneous differential equation

$$y''' - 8y = -16x$$

(Hint: $r = 2$ is a root of the characteristic polynomial.)

Characteristic polynomial

$$p(r) = r^3 - 8 = (r-2)(r^2 + 2r + 4)$$

whose roots are

$$r_1 = 2, r_{2,3} = -1 \pm i\sqrt{3}$$

Hence the homogeneous solution is

$$y_h(x) = C_1e^{2x} + e^{-x}\left(C_2\cos(\sqrt{3}x) + C_3\sin(\sqrt{3}x)\right)$$

The non-homogeneous term $-16x$ *is not contained in the homogeneous solution so we look for a particular solution of the form* $y_p = A + Bx$*. By substitution, we find* $B = 2$*, hence the general solution is*

$$y(x) = C_1e^{2x} + e^{-x}\left(C_2\cos(\sqrt{3}x) + C_3\sin(\sqrt{3}x)\right) + 2x$$

(5)[3Pts] Consider the following differential equations modeling *harmonic motion with forced vibration.* For each equation, explain if (i) the equation models a damped or undamped system and if (ii) the solution is in resonance (this occurs when the amplitude of vibrations increases without bound) or not.

(a) $y'' + 4y = \sin(x)$

(b) $y'' + 4y = \sin(2x)$

(c) $y'' + 2y' + 3y = \sin(2x)$

(a-b) They model an **undamped** *system, since there is no first derivative term modeling friction. Characteristic polynomial*

$$p(r) = r^2 + 4$$

whose roots are $r = \pm 2i$. *Hence the homogeneous solution is*

$$y_h(x) = C_1 \cos(2x) + C_2 \sin(2x) = A \sin(2x + \phi)$$

(a) The forcing term is NOT contained in the homogeneous solution, hence there is **no resonance**.
(b) The forcing term is contained in the homogeneous solution so there is **resonance**. *In fact, a particular solution of the form* $y_p(x) = Bx \sin(2x + \psi)$
(c) It models a **damped** *system, since there is a first derivative term modeling friction. Characteristic polynomial:*

$$p(r) = r^2 + 2r + 3$$

whose roots are $r = -1 \pm i\sqrt{2}$. *Hence the homogeneous solution is*

$$y_h(x) = e^{-x}(C_1 \cos(\sqrt{2}x) + C_2 \sin(\sqrt{2}x) = Ae^{-x} \sin(\sqrt{2}x + \phi)$$