Math 3333 Name:

HW 10

Please, write clearly and justify all your statements using the material covered in class to get credit for your work.

(1) Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Use the chain rule and the product rule to show that f is differentiable at each $x \neq 0$ and find $f'(x)$.
(b) Use the definition to show that f is differentiable at $x = 0$ and find $f'(0)$.
(c) Show that f' is not continuous at $x = 0$.
(d) Let $g(x) = x^2$ if $x \leq 0$ and $g(x) = x^2 \sin \frac{1}{x}$ if $x > 0$. Determine whether or not g is differentiable at $x = 0$. If it is, find $g'(0)$.

(2) Let $f(x) = x^2$ if x is rational and $f(x) = 0$ if x is irrational.
(a) Prove that f is continuous at exactly one point, namely at $x = 0$.
(b) Prove that f is differentiable at exactly one point, namely at $x = 0$.

(3) Use the mean value theorem to establish the following inequalities
(a) $e^x > 1 + x$, for $x > 0$.
(b) $\frac{x-1}{x} < \ln x < x - 1$, for $x > 1$.
(f) $\sin x \leq x$, for $x \geq 0$.

(4) A differentiable function f is said to be increasing on an interval I if $x_1 < x_2$ in I implies that $f(x_1) \leq f(x_2)$. Prove that f is increasing on I iff $f'(x) \geq 0$ for all $x \in I$.