Math 3333 Name: **SOLUTION**

HW 1

Please, write clearly and justify your arguments using the theory covered in class to get credit for your work.

(1) [3Pts] Prove that

$$\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1) \quad n \in \mathbb{N}.$$

Solution: For $n = 1$ it is easily verified.
Now assume it works for $n = k$.
For $n = k + 1$, we have

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 \\
&= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\
&= \frac{1}{6}(k+1)\left(k(2k+1) + 6(k+1)\right) \\
&= \frac{1}{6}(k+1)\left(2k^2 + 7k + 6\right) \\
&= \frac{1}{6}(k+1)(k+2)(2k+3) \\
&= \frac{1}{6}(k+1)\left((k+1)+1\right)\left(2(k+1)+1\right) \quad \textbf{QED.}
\end{aligned}$$

(2) [3Pts] Prove that, for any $n \in \mathbb{N}$, the number $9^n - 4^n$ is divisible by 5.

Solution: For $n = 1$, 9-4 =5, hence it is true.
Now assume it is true for $n = k$; that is, there is an $m \in \mathbb{N}$ such that $9^k - 4^k = 5m$.
So, for $n = k + 1$ we get

$$\begin{aligned}
9^{k+1} - 4^{k+1} &= 9 \cdot 9^k - 4 \cdot 4^k \\
&= 9(9^k - 4^k) + 9 \cdot 4^k - 4 \cdot 4^k \\
&= 9 \cdot 5 \cdot m + 5 \cdot 4^k \\
&= 5(9 \cdot m + 4^k) \quad \textbf{QED.}
\end{aligned}$$

(3) [3Pts] Prove that, for any $n \geq 4$ the following inequality holds

$$n^2 \leq 2^n.$$

Solution: It satisfies for $n = 4$ since $4^2 = 2^4 = 16$.
Now assume it is true for $n = k > 4$. That is, $k^2 \leq 2^k$.
Then for $n = k + 1$ we have

$$(k+1)^2 = k^2 + 2k + 1 \leq 2^k + 2k + 1.$$

We observe [1] that, for $k > 4$, $2k + 1 < 2k + k < 3k < k^2$. Hence, going back to the prior inequality and using the inductive step we have

$$(k+1)^2 = k^2 + 2k + 1 \leq k^2 + k^2 \leq 2^k + 2^k = 2^{k+1} \quad \textbf{QED}.$$

(4) [1Pts] Let $x, y \in \mathbb{R}$ and $\epsilon > 0$. Prove that is $|x-y| \leq \epsilon$, then $|x| \leq |y| + \epsilon$.

Solution: Using the triangle inequality, we have that

$$|x| = |x - y + y| \leq |x - y| + |y| \leq \epsilon + |y| \quad \textbf{QED}.$$

[1] Alternative argument: We can show that $2k + 1 < k^2$ for $k > 0$ by observing that the quadratic equation $k^2 - 2k - 1$ has roots at $1 \pm \sqrt{2}$ hence its positive for $k > 1 + \sqrt{2}$.