HW 2

Please, write clearly and justify your arguments using the theory covered in class to get credit for your work.

(1) [3Pts] Let S, T be nonempty subsets of $\mathbb{R}$ and suppose that $S \subset T$. Prove that

$$\inf T \leq \inf S \leq \sup S \leq \sup T$$

Proof. For every $t \in T$, by definition it is $\inf T \leq t$.

Since $S \subset T$, then for every $s \in S$, it is $\inf T \leq s$. This shows that $\inf T$ is a lower bond of S, hence $\inf T \leq \inf S$.

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For every $t \in T$, by definition it is $t \leq \sup T$. Since $S \subset T$ then $s \leq \sup T$ for all $S \in S$. This shows that $\sup T$ is an upper bound of S, hence $\sup S \leq \sup T$.

Combining these observations, we conclude that

$$\inf T \leq \inf S \leq \sup S \leq \sup T.$$

(2) [3Pts] Let S be a nonempty and bounded subset of $\mathbb{R}$. Prove that $M = \sup S$ is unique.

Proof. Suppose that there exists another number $M_1 = \sup S$ with $M_1 \neq M$. Then either $M_1 > M$ or $M_1 < M$. If $M_1 > M$ then M_1 would not be $\sup S$ since it could not be the least upper bound of S. Similarly, if $M_1 < M$ then M would not be $\sup S$ since it could not be the least upper bound of S. Thus it must be $M = M_1$

(3) [3Pts] Let $S = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$. Prove that $\sup S = 1$ and find the accumulation points of S is any. Justify your answer.

Proof. $1 - \frac{1}{n} \leq 1$, for all n, hence 1 is an upper bound of S. To show that 1 is the least upper bound, observe that, if $M = 1 - \epsilon$, for some $\epsilon > 0$, by the Archimedean property there exists some $n \in \mathbb{N}$ such that $1 - \frac{1}{n} > 1 - \epsilon$, so that M cannot be an upper bound. Hence $\sup S = 1$.

1 is an accumulation point of S since, for any interval of the form $(1-\epsilon, 1+\epsilon)$, with $\epsilon > 0$, the Archimedean property implies that there exists some $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$ so that $1+\epsilon > 1-\frac{1}{n} > 1-\epsilon$. S has no other accumulation points. For any point $x_n = 1-\frac{1}{n}$, the distance to the closest point is $\frac{1}{n(n+1)}$, so that the deleted neighborhood $N(x_n, r_n)$ with $r_n < \frac{1}{n(n+1)}$ has empty intersection with the set S.

(4) [3Pts] Let $X \in \mathbb{R}$ be nonempty and f, g be bounded functions defined on X. Prove that

$$\sup\{f(x)+g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}.$$

Give examples to show that the inequality can be either an equality or a strict inequality.

Proof. *For any $x \in X$, $f(x) \leq \sup\{f(y) : y \in X\}$ and $g(x) \leq \sup\{g(y) : y \in X\}$. Hence for any $x \in X$,*

$$f(x)+g(x) \leq \sup\{f(y) : y \in X\} + g(x) \leq \sup\{g(y) : y \in X\}.$$

This shows that the right hand side is an upper bound of the set $\{f(x)+g(x) : x \in X\}$. Hence

$$\sup\{f(x)+g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}.$$

Example (equality). Set $f(x) = x$, $g(x) = 1$, for $x \in [0,1]$ Then

$$\begin{aligned} 2 &= \sup\{f(x)+g(x) : x \in [0,1]\} \\ &= \sup\{f(x) : x \in [0,1]\} + \sup\{g(x) : x \in [0,1]\} \\ &= 1+1. \end{aligned}$$

Example (inequality). Set $f(x) = x$, $g(x) = -x$, for $x \in [0,1]$ Then

$$\begin{aligned} 0 &= \sup\{f(x)+g(x) : x \in [0,1]\} \\ &= \sup\{f(x) : x \in [0,1]\} + \sup\{g(x) : x \in [0,1]\} \\ &= 1+0. \end{aligned}$$

(5) [3Pts] Let $S \subset \mathbb{R}$ be nonempty. Show that S is bounded if and only if there exists a closed bounded interval I such that $S \subset I$.

Proof. If S is bounded then let $m = \inf S$ and $M = \sup S$. It follows that S is contained in the interval $I = [m, M]$.

Conversely, suppose that $S \subset I = [a,b]$, where $a, b \in \mathbb{R}$. It follows that $a \leq \inf S$ and $\sup S \leq b$. Hence S is bounded.