**Math 3333** Name: SOLUTION

## HW 3

Please, write clearly and justify your arguments using the theory covered in class to get credit for your work.

(1) [4 Pts] Prove the following.

(a) An accumulation point of a set $S$ is either an interior point of $S$ or a boundary point of $S$.

*Proof.* *If $x$ is an interior point of $S$ then it is an accumulation point of $S$ since any deleted neighborhood $N^*(x, \epsilon)$ contains elements of $S$. Next, let $x$ be an accumulation point of set $S$ and suppose that $x$ is not an interior point of $S$. Let $N(x, \epsilon)$ be an arbitrary neighborhood of $x$. Since $x$ is not an interior point of $S$ then $N(x, \epsilon)$ is not contained in $S$, that is, $N(x, \epsilon) \cap (R \setminus S) \neq \emptyset$. On the other hand, since $x$ is an accumulation point of $S$ then every deleted neighborhood of $x$ and hence every neighborhood of $x$ must contain a point of $S$. Thus $N(x, \epsilon) \cap S \neq \emptyset$. It follows that $x$ is a boundary point of $S$.*

(b) A boundary point of a set $S$ is either an accumulation point of $S$ or an isolated point of $S$.

*Proof.* *Let $x \in \text{bd}\, S$ and suppose $x \in S$. If $x$ is not an accumulation point of $S$ then by definition $x$ is an isolated point of $S$. On the other hand, suppose $x \notin S$. Since $x \in \text{bd}\, S$, every neighborhood of $x$ must intersect $S$ in a point, but this point cannot be $x$ since $x \notin S$. This means every deleted neighborhood of $x$ must contain a point of $S$ hence $x$ is an accumulation point of $S$.*

(2) [5 Pts] Mark each statement as True or False. If False, show a counter-example. If True, justify your answer.

(a) Every finite set is compact.
*True. By theorem discussed in class.*

(b) The set $S = \{\frac{1}{n} : n \in \mathbb{N}\}$ is compact.
*False. The set $S$ is not closed since $0$ is an accumulation point of $S$ but $0 \notin S$.*

(c) If $S$ is unbounded then $S$ has an accumulation point.
*False. The set $\mathbb{N}$ is unbounded yet has no accumulation points.*

(d) If $S \subset \mathbb{R}$ is compact and $x$ is an accumulation point of $S$, then $x \in S$.
*True. If a set is compact then it is closed and it must contain all its accumulation points.*

(e) If $S \subset \mathbb{R}$ is a compact, then there is at least one point in $\mathbb{R}$ that is an accumulation point of $S$.

*False. The set* $S = \{1, 2, 3\}$ *is compact since closed and bounded but it contains no accumulation points.*

(3) [3 Pts] Prove or give a counterexample: If a set $S$ has a maximum and a minimum, then $S$ is a closed set.

*This is false. Consider the set* $S = [1, 2) \cup (3, 4]$. *This set has a maximum and a minimum (min S = 1, max S = 4) but the set is clearly not closed.*

(4) [4 Pts]

(a) Let $S_1, S_2$ be compact subsets of $\mathbb{R}$. Prove that $S_1 \cup S_2$ is also compact.

(b) Find an infinite collection of compact subsets $\{S_n : n \in \mathbb{N}\}$ such that the union $\cup_n S_n$ is not compact. Explain why the resulting set is not compact.

*(a) Proof. Since* $S_1$ *and* $S_2$ *are compact they are closed and bounded sets (by the Heine-Borel theorem). Hence* $S_1 \cup S_2$ *is a closed set since is a finite union of closed sets.* $S_1 \cup S_2$ *is also bounded since* $\sup(S_1 \cup S_2) \le \sup S_1 + \sup S_2$ *and* $\inf(S_1 \cup S_2) \ge \inf S_1 + \inf S_2$. *It follows that* $S_1 \cup S_2$ *is compact (by the Heine-Borel theorem).*

*(b) Let* $S_n = [-n, n]$. *Each set* $S_n$ *is compact since closed and bounded. However,* $\cup_n S_n = \mathbb{R}$ *and this set is not compact.*

(5) [3 Pts] Using the definition of compactness and the fact that a compact set is closed, prove that the intersection of any collection of compact subsets is also compact.

*Proof. If* $\cap S_n = \varnothing$ *then the statement is trivially true. Hence assume that* $\cap S_n \neq \varnothing$. *Let* $(S_n)$ *be a collection of compact sets and* $O$ *be an open cover of* $G = \cap S_n$. *Since* $\cap S_n$ *is closed, then* $G^c$ *is open. Let* $O' = O \cup G^c$. $O'$ *is an open cover of* $S_1$ *and, since* $S_1$ *is compact, it has a finite open subcover* $F \subset O'$. *Since* $G = \cap S_n \subset S_1$, *then* $F$ *is also a finite open cover of* $G$ *and* $F \setminus G^c \subset O$ *is also finite open cover of* $G = \cap S_n$. *This shows that any open cover* $O$ *of* $G$ *has a finite open subcover.*

(5b) [3 Pts] Prove that the intersection of any collection of compact subsets is also compact.

*Proof. Let* $(S_n)$ *be a collection of compact sets. Each set* $S_n$ *is a closed and bounded set (by the Heine-Borel theorem). It follows by the properties of closed sets that* $\cap S_n$ *is a closed set.* $\cap S_n$ *is also bounded since* $\sup(\cap S_n) \le$

$\sup_n(\max S_n)$ *and* $\inf(\cap S_n) \geq \inf_n(\min S_n)$. *It follows that* $\cap S_n$ *is compact (by the Heine-Borel theorem).*