Math 3333 Name: SOLUTION

HW 3

Please, write clearly and justify your arguments using the theory covered in class to get credit for your work.

(1) [4 Pts] Prove the following.

(a) An accumulation point of a set S is either an interior point of S or a boundary point of S.

Proof. *If x is an interior point of S then it is an accumulation point of S since any deleted neighborhood $N^*(x,\epsilon)$ contains elements of S. Next, let x be an accumulation point of set S and suppose that x is not an interior point of S. Let $N(x,\epsilon)$ be an arbitrary neighborhood of x. Since x is not an interior point of S then $N(x,\epsilon)$ is not contained in S, that is, $N(x,\epsilon) \cap (R \setminus S) \neq \emptyset$. On the other hand, since x is an accumulation point of S then every deleted neighborhood of x and hence every neighborhood of x must contain a point of S. Thus $N(x,\epsilon) \cap S \neq \emptyset$. It follows that x is a boundary point of S.*

(b) A boundary point of a set S is either an accumulation point of S or an isolated point of S.

Proof. *Let $x \in \text{bd}\, S$ and suppose $x \in S$. If x is not an accumulation point of S then by definition x is an isolated point of S. On the other hand, suppose $x \notin S$. Since $x \in \text{bd}\, S$, every neighborhood of x must intersect S in a point, but this point cannot be x since $x \notin S$. This means every deleted neighborhood of x must contain a point of S hence x is an accumulation point of S.*

(2) [5 Pts] Mark each statement as True or False. If False, show a counterexample. If True, justify your answer.

(a) Every finite set is closed.

True. If a set S is finite, then each point is isolated since, for each $x \in S$ we can find an $\epsilon > 0$ such that $N^*(x,\epsilon) \cap S = \emptyset$. (If this was not the case, then the set would not be finite). Hence S contains all its boundary points and must be closed.

(b) The set $\{\frac{1}{n} : n \in \mathbb{N}\}$ has no accumulation points.

False. 0 is an accumulation point of S but $0 \notin S$.

(c) If S is unbounded then S has an accumulation point.

False. The set $\mathbb{N}$ is unbounded yet has no accumulation points.

(d) If $S \subset \mathbb{R}$ is open and x is an accumulation point of S, then $x \in S$.

False. The set $S = (0,1)$ *is open and* $x = 0$ *is an accumulation point of* S *but* $x \notin S$.

(e) If $S \subset \mathbb{R}$ is a closed, then there is at least one point in $\mathbb{R}$ that is an accumulation point of S.

False. The set $S = \{1, 2, 3\}$ *is closed and bounded but it contains no accumulation points.*

(3) [3 Pts] Prove or give a counterexample: If a set S has a maximum and a minimum, then S is a closed set.

This is false. Consider the set $S = [1, 2) \cup (3, 4]$. *This set has a maximum and a minimum (min S = 1, max S = 4) but the set is clearly not closed.*

(4) [4 Pts]

(a) Let S_1, S_2 be closed subsets of $\mathbb{R}$. Prove that $S_1 \cup S_2$ is also closed.

(b) Find an infinite collection of closed subsets $\{S_n : n \in \mathbb{N}\}$ such that the union $\cup_n S_n$ is not closed. Explain why the resulting set is not closed.

(a) Proof. The union of two closed sets $S_1 \cup S_2$ *is closed iff* $(S_1 \cup S_2)^c = S_1^c \cap S_2^c$ *is open. hence it will be sufficient to prove that the intersection of two open sets* A_1, A_2 *is open. Take* $x \in A_1 \cap A_2$. *Since the sets* A_1, A_2 *are open, there are* $\epsilon_1, \epsilon_2 >$ *such that* $N(x, \epsilon_1) \subset A_1$ *and* $N(x, \epsilon_2) \subset A_2$. *Set* $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. *Then* $N(x, \epsilon) \subset A_1$ *and* $N(x, \epsilon_2) \subset A_1$, *hence* $N(x, \epsilon_2) \subset A_1 \cap A_2$. *This shows that the set* $A_1 \cap A_2$ *is open.*

(b) Let $S_n = [\frac{1}{n}, \infty)$. *Each set* S_n *is closed. However,* $\cup_n S_n = (0, \infty)$ *and this set is open.*