Math 3333 Name: SOLUTION

HW 4

Please, write clearly and justify all your statements using the material covered in class to get credit for your work.

(1) [4 Pts] Use the definition of convergence to prove the following:

(a) For any real number k, $\lim_{n\to\infty} k/n = 0$

We need to show that, given $\epsilon > 0$, there exists $N = N(\epsilon)$ such that

$$\left|\frac{k}{n}\right| < \epsilon$$

provided $n > N$. For that, let $N = \lceil \frac{|k|}{\epsilon} \rceil$. Then for all $n > N$ we have that $|\frac{k}{n}| < \frac{|k}{N} < \epsilon$.

(b) $\lim_{n\to\infty} \frac{3n+1}{n+2} = 3$.

We need to show that, given $\epsilon > 0$, there exists $N = N(\epsilon)$ such that

$$\left|\frac{3n+1}{n+2} - 3\right| = \frac{3}{n+2} < \epsilon$$

provided $n > N$. For that, choose, $N = \lceil \frac{3}{\epsilon} \rceil$. Then $\frac{3}{n+2} < \frac{3}{n} < \epsilon$ if $n > N$.

(2) [3 Pts] Show that the sequence $a_n = cos\frac{n\pi}{3}$ is divergent.

Arguing by contradiction, suppose that $\lim a_n = a$. It then follows by definition that there exists an $N \in \mathbb{N}$ such that

$$\left|cos\frac{n\pi}{3} - a\right| < 1, \quad \textit{for all } n > N.$$

If we take $n = 6m$, then the inequality above implies that $|cos(2m\pi) - a| < 1$, that is $|1 - a| < 1$ so that $0 < a < 2$. If instead we take $n = 3(2m-1)$, then the inequality above implies that $|cos((2m-1)\pi) - a| < 1$, that is $|1 + a| < 1$ so that $-2 < a < 0$. Since the two conditions on a cannot be satisfied at the same time, then we have a contradiction.

(3) [3 Pts]

(a) Let (s_n) be a sequence such that $\lim_{n\to\infty} s_n = 0$ and (t_n) be a bounded sequence. Prove that the sequence $(s_n\, t_n)$ is convergent.

(b) Show by example that the boundedness of (t_n) is necessary in part (a). That is, produce an example to show that the sequence $(s_n\, t_n)$ may diverge if (t_n) is not bounded.

(a) *Proof. Since* (t_n) *is bounded, there is an* $M > 0$ *such that* $t_n < M$ *for all* $n \in \mathbb{N}$. *Since* $\lim_{n\to\infty} s_n = 0$, *given any* $\epsilon > 0$, *there exists and* $N = N(\epsilon)$ *such that* $|s_n| < \frac{\epsilon}{M}$ *if* $n > N$. *It follows that, given* $\epsilon > 0$, *there exists* $N = N(\epsilon)$ *such that* $|s_n\, t_n| < \frac{\epsilon}{M} M = \epsilon$ *if* $n > N$.

(b) *Consider the sequences* $(s_n) = (\frac{1}{n})$ *and* $(t_n) = (n^2)$. *Then* $(s_n\, t_n) = (n)$ *and this sequence is not convergent.*

(4)[3 Pts] Prove or give a counterexamples:

(a) If (s_n) and (t_n) are divergent sequences, then $(s_n + t_n)$ diverges.
FALSE. Let $(s_n) = (-1)^n$ *and* $(t_n) = (-1)^{n+1}$. $(s_n + t_n) = 0$ *convergent.*

(b) If (s_n) and (t_n) are divergent sequences, then $(s_n\, t_n)$ diverges.
FALSE. Let $(s_n) = (-1)^n$ *and* $(t_n) = (-1)^n$. $(s_n\, t_n) = 1$ *convergent.*

(c) If (s_n) and $(s_n + t_n)$ are convergent sequences, then (t_n) converges.
TRUE by Limit Theorems. $(t_n) = (s_n + t_n) - (s_n)$ *convergent since it is an algebraic sum of convergent sequences.*

(5)[3 Pts] Prove that if (x_n) is a convergent sequence, $(|x_n|)$ is also convergent. Is the converse true?

Proof.

Since (x_n) *converges,* $\lim x_n = s$. *Hence, given any* $\epsilon > 0$, *there exists an* $N = N(\epsilon)$ *such that* $|x_n - s| < \epsilon$ *if* $n > N$.

Since $|x_n| \leq |x_n - s| + |s|$ *and* $|s| \leq |s - x_n| + |x_n|$, *it follows that* $||x_n| - |s|| \leq |x_n - s|$. *It follows that* $||x_n| - |s|| < \epsilon$ *if* $n > N$. *Hence* $(|x_n|)$ *converges and* $\lim |x_n| = |s|$.

The converse is not true. Consider $(x_n) = (-1)^n$. *In this case,* $(|x_n|) = 1$ *is convergent but* (x_n) *is not convergent.*

(6)[3 Pts] Suppose that (x_n) is a convergent sequence and (y_n) is a sequence such that, for any $\epsilon > 0$, there exists an $M > 0$ such that $|x_n - y_n| < \epsilon$ for all $n > M$. Does it follow that (y_n) converge? Prove it or find a counterexample.

Proof.

Since (x_n) *converges,* $\lim x_n = s$. *Hence, given any* $\epsilon > 0$, *there exists an* $N_1 = N_1(\epsilon)$ *such that* $|x_n - s| < \epsilon$ *if* $n > N_1$. *Since, for any* $n \in \mathbb{N}$,

$$|y_n - s| = |y_n - x_n + x_n - s| \leq |y_n - x_n| + |x_n - s|,$$

it follows that $|y_n - s| < 2\epsilon$ *if* $n > N = \max\{N_1, M\}$. *This proves that* $\lim y_n = s$.