Math 3333 Name:

HW 4

Please, write clearly and justify all your statements using the material covered in class to get credit for your work.

(1) [4 Pts] Mark each statement as True or False. If False, show a counter-example. If True, justify your answer.

(a) Every finite set is compact.
(b) The set $\{\frac{1}{n} : n \in \mathbb{N}\}$ is compact.
(c) If $S \subset \mathbb{R}$ is compact and x is an accumulation point of S, then $x \in S$.
(d) If $S \subset \mathbb{R}$ is a compact, then there is at least one point in $\mathbb{R}$ that is an accumulation point of S.

(2) [6 Pts]
(a) Let S_1, S_2 be compact subsets of $\mathbb{R}$. Prove that $S_1 \cup S_2$ is also compact.
(b) Find an infinite collection of compact subsets $\{S_n : n \in \mathbb{N}\}$ such that the union $\cup_n S_n$ is not compact. Explain why the resulting set is not compact.
(c) Using the definition of compactness, prove that the intersection of any collection of compact subsets is also compact.

(3) [4 Pts] Use the definition of convergence to prove the following:
(a) For any real number k, $\lim_{n\to\infty} k/n = 0$
(b) $\lim_{n\to\infty} \frac{3n+1}{n+2} = 3$.

(4) [3 Pts] Show that the sequence $a_n = cos\frac{n\pi}{3}$ is divergent.

(5) [4 Pts]
(a) Let (s_n) be a sequence such that $\lim_{n\to\infty} s_n = 0$ and (t_n) be a bounded sequence. Prove that the sequence $(s_n\, t_n)$ is convergent.
(b) Show by example that the boundedness of (t_n) is necessary in part (a). That is, produce an example to show that the sequence $(s_n\, t_n)$ may diverge if (t_n) is not bounded.

(6)[6 Pts] Prove or give a counterexamples:
(a) If (s_n) and (t_n) are divergent sequences, then $(s_n + t_n)$ diverges.
(b) If (s_n) and (t_n) are divergent sequences, then $(s_n\, t_n)$ diverges.
(c) If (s_n) and $(s_n + t_n)$ are convergent sequences, then (t_n) converges.

(7)[2 Pts] Prove that if (x_n) is a convergent sequence, $(|x_n|)$ is also convergent. Is the converse true?

(8)[3 Pts] Suppose that (x_n) is a convergent sequence and (y_n) is a sequence such that, for any $\epsilon > 0$, there exists an $M > 0$ such that $|x_n - y_m| < \epsilon$ for all $n > M$. Does it follow that (y_n) converge? Prove it or find a counterexample.