Math 3333 Name: SOLUTION

HW 4

Please, write clearly and justify all your statements using the material covered in class to get credit for your work.

(1) [4 Pts] Mark each statement as True or False. If False, show a counterexample. If True, justify your answer.

(a) Every finite set is compact.
True. By theorem discussed in class.

(b) The set $\{\frac{1}{n} : n \in \mathbb{N}\}$ is compact.
False. The set S is not closed since 0 is an accumulation point of S but $0 \notin S$.

(c) If $S \subset \mathbb{R}$ is compact and x is an accumulation point of S, then $x \in S$.
True. If a set is compact then it is closed and it must contain all its accumulation points.

(d) If $S \subset \mathbb{R}$ is a compact, then there is at least one point in $\mathbb{R}$ that is an accumulation point of S.
False. The set $S = \{1, 2, 3\}$ is compact since closed and bounded but it contains no accumulation points.

(2) [6 Pts]

(a) Let S_1, S_2 be compact subsets of $\mathbb{R}$. Prove that $S_1 \cup S_2$ is also compact.

(b) Find an infinite collection of compact subsets $\{S_n : n \in \mathbb{N}\}$ such that the union $\cup_n S_n$ is not compact. Explain why the resulting set is not compact.

(c) Using the definition of compactness, prove that the intersection of any collection of compact subsets is also compact.

(a) Proof. Since S_1 and S_2 are compact they are closed and bounded sets (by the Heine-Borel theorem). Hence $S_1 \cup S_2$ is a closed set since is a finite union of closed sets. $S_1 \cup S_2$ is also bounded since $\sup(S_1 \cup S_2) \leq \sup S_1 + \sup S_2$ and $\inf(S_1 \cup S_2) \geq \inf S_1 + \inf S_2$. It follows that $S_1 \cup S_2$ is compact (by the Heine-Borel theorem).

(b) Let $S_n = [-n, n]$. Each set S_n is compact since closed and bounded. However, $\cup_n S_n = \mathbb{R}$ and this set is not compact.

(c) Proof. Let (S_n) be a collection of compact sets. Each set S_n is a closed and bounded set (by the Heine-Borel theorem). It follows by the properties of closed sets that $\cap S_n$ is a closed set. $\cap S_n$ is also bounded since $\sup(\cap S_n) \leq \sup_n(\max S_n)$ and $\inf(\cap S_n) \geq \inf_n(\min S_n)$. It follows that $\cap S_n$ is compact (by the Heine-Borel theorem).

(3) [4 Pts] Use the definition of convergence to prove the following:
(a) For any real number k, $\lim_{n\to\infty} k/n = 0$
We need to show that, given $\epsilon > 0$, there exists $N = N(\epsilon)$ such that

$$\left|\frac{k}{n}\right| < \epsilon$$

provided $n > N$. For that, let $N = \lceil \frac{|k|}{\epsilon} \rceil$. Then for all $n > N$ we have that $|\frac{k}{n}| < \frac{|k}{N} < \epsilon$.

(b) $\lim_{n\to\infty} \frac{3n+1}{n+2} = 3$.
We need to show that, given $\epsilon > 0$, there exists $N = N(\epsilon)$ such that

$$\left|\frac{3n+1}{n+2} - 3\right| = \frac{3}{n+2} < \epsilon$$

provided $n > N$. For that, choose, $N = \lceil \frac{3}{\epsilon} \rceil$. Then $\frac{3}{n+2} < \frac{3}{n} < \epsilon$ if $n > N$.

(4) [3 Pts] Show that the sequence $a_n = cos\frac{n\pi}{3}$ is divergent.

Arguing by contradiction, suppose that $\lim a_n = a$. It then follows by definition that there exists an $N \in \mathbb{N}$ such that

$$\left|cos\frac{n\pi}{3} - a\right| < 1, \quad \text{for all } n > N.$$

If we take $n = 6m$, then the inequality above implies that $|cos(2m\pi) - a| < 1$, that is $|1 - a| < 1$ so that $0 < a < 2$. If instead we take $n = 3(2m-1)$, then the inequality above implies that $|cos((2m-1)\pi) - a| < 1$, that is $|1 + a| < 1$ so that $-2 < a < 0$. Since the two conditions on a cannot be satisfied at the same time, then we have a contradiction.

(5) [3 Pts]
(a) Let (s_n) be a sequence such that $\lim_{n\to\infty} s_n = 0$ and (t_n) be a bounded sequence. Prove that the sequence $(s_n\, t_n)$ is convergent.
(b) Show by example that the boundedness of (t_n) is necessary in part (a). That is, produce an example to show that the sequence $(s_n\, t_n)$ may diverge if (t_n) is not bounded.
(a) *Proof. Since (t_n) is bounded, there is an $M > 0$ such that $t_n < M$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} s_n = 0$, given any $\epsilon > 0$, there exists and $N = N(\epsilon)$ such that $|s_n| < \frac{\epsilon}{M}$ if $n > N$. It follows that, given $\epsilon > 0$, there exists $N = N(\epsilon)$ such that $|s_n\, t_n| < \frac{\epsilon}{M} M = \epsilon$ if $n > N$.*

(b) *Consider the sequences* $(s_n) = (\frac{1}{n})$ *and* $(t_n) = (n^2)$. *Then* $(s_n\, t_n) = (n)$ *and this sequence is not convergent.*

(6)[3 Pts] Prove or give a counterexamples:

(a) If (s_n) and (t_n) are divergent sequences, then $(s_n + t_n)$ diverges.

FALSE. Let $(s_n) = (-1)^n$ *and* $(t_n) = (-1)^{n+1}$. $(s_n + t_n) = 0$ *convergent.*

(b) If (s_n) and (t_n) are divergent sequences, then $(s_n\, t_n)$ diverges.

FALSE. Let $(s_n) = (-1)^n$ *and* $(t_n) = (-1)^n$. $(s_n\, t_n) = 1$ *convergent.*

(c) If (s_n) and $(s_n + t_n)$ are convergent sequences, then (t_n) converges.

TRUE by Limit Theorems. $(t_n) = (s_n + t_n) - (s_n)$ *convergent since it is an algebraic sum of convergent sequences.*

(7)[3 Pts] Prove that if (x_n) is a convergent sequence, $(|x_n|)$ is also convergent. Is the converse true?

Proof.

Since (x_n) *converges,* $\lim x_n = s$. *Hence, given any* $\epsilon > 0$, *there exists an* $N = N(\epsilon)$ *such that* $|x_n - s| < \epsilon$ *if* $n > N$.

Since $|x_n| \leq |x_n - s| + |s|$ *and* $|s| \leq |s - x_n| + |x_n|$, *it follows that* $||x_n| - |s|| \leq |x_n - s|$. *It follows that* $||x_n| - |s|| < \epsilon$ *if* $n > N$. *Hence* $(|x_n|)$ *converges and* $\lim |x_n| = |s|$.

The converse is not true. Consider $(x_n) = (-1)^n$. *In this case,* $(|x_n|) = 1$ *is convergent but* (x_n) *is not convergent.*

(8)[3 Pts] Suppose that (x_n) is a convergent sequence and (y_n) is a sequence such that, for any $\epsilon > 0$, there exists an $M > 0$ such that $|x_n - y_n| < \epsilon$ for all $n > M$. Does it follow that (y_n) converge? Prove it or find a counterexample.

Proof.

Since (x_n) *converges,* $\lim x_n = s$. *Hence, given any* $\epsilon > 0$, *there exists an* $N_1 = N_1(\epsilon)$ *such that* $|x_n - s| < \epsilon$ *if* $n > N_1$. *Since, for any* $n \in \mathbb{N}$,

$$|y_n - s| = |y_n - x_n + x_n - s| \leq |y_n - x_n| + |x_n - s|,$$

it follows that $|y_n - s| < 2\epsilon$ *if* $n > N = \max\{N_1, M\}$. *This proves that* $\lim y_n = s$.