Math 3333 Name: SOLUTION

HW 5

Please, write clearly and justify all your statements using the material covered in class to get credit for your work.

(1) Prove that

$$\lim_{n\to\infty} \sqrt{n^2+1} - n = 0$$

Proof. *Observe that*

$$|\sqrt{n^2+1} - n| = \frac{1}{\sqrt{n^2+1}+n} < \frac{1}{n}$$

Given any $\epsilon > 0$, *let* $N > \frac{1}{\epsilon}$, *then*

$$|\sqrt{n^2+1} - n| < \frac{1}{n} < \epsilon, \quad \text{if } n > N.$$

This proves that $\lim_{n\to\infty} \sqrt{n^2+1} - n = 0$.

(2) Prove that if $\lim_{n\to\infty} s_n = \infty$ and if (t_n) is a bounded sequence, then

$$\lim_{n\to\infty} (s_n + t_n) = \infty$$

Proof. *Since* (t_n) *is bounded, there exists an* $N_1 \in \mathbb{N}$ *and a* $L > 0$ *such that* $|t_n| < L$ *if* $n > N_1$. *That is,* $-L < t_n < L$ *if* $n > N_1$.

Since $\lim_{n\to\infty} s_n = \infty$, *given any* $M > 0$, *there exists an* N_2 *such that* $s_n > M + L$ *if* $n > N_2$.

Hence, provided $n > \max\{N_1, N_2\}$, *we have that* $s_n + t_n > M$. *Since* M *is arbitrary, this proves that* $\lim_{n\to\infty}(s_n + t_n) = \infty$.

(3) Prove that if $\lim_{n\to\infty} s_n = \infty$ and $\lim_{n\to\infty} t_n = L > 0$, then

$$\lim_{n\to\infty} (s_n t_n) = \infty$$

Proof. *Since* $\lim_{n\to\infty} t_n = L$, *there exists an* N_1 *such that* $|t_n - L| < L/2$ *if* $n > N_1$. *Hence,* $t_n > L/2$, *is* $n > N_1$.

Since $\lim_{n\to\infty} s_n = \infty$, *given any* $M > 0$, *there exists an* N_2 *such that* $s_n > 2M/L$ *if* $n > N_2$.

Hence, provided $n > \max N_1, N_2$, *we have that* $s_n t_n > M$. *Since* M *is arbitrary, this proves that* $\lim_{n\to\infty}(s_n t_n) = \infty$.

(4) Prove that the sequence below is monotone and bounded. Next find its limit.

$$s_1 = 1, \quad s_{n+1} = \frac{1}{5}(s_n + 7), \quad n \geq 1.$$

Proof. Note that $s_1 = 1$, $s_2 = \frac{1}{5}(1+7) = \frac{8}{5}$.

Claim: $s_n \leq 2$. *Proof by induction:*

$s_1 = 1 < 2$.

Assume $s_n \leq 2$.

Then $s_{n+1} = \frac{1}{5}(s_n + 7) \leq s_{n+1} = \frac{1}{5}(2+7) = \frac{9}{5} < 2$.

Claim: $s_{n+1} \geq s_n$. *Proof by induction:*

$s_2 < s_1$.

Assume $s_{n+1} \geq s_n$.

Then $s_{n+2} = \frac{1}{5}(s_{n+1} + 7) \geq \frac{1}{5}(s_n + 7) = s_{n+1}$.

Since (s_n) *is monotone and bounded, then it is convergent. Thus*

$$s = \lim s_{n+1} = \lim \frac{1}{5}(s_n + 7) = \frac{1}{5}(s + 7).$$

Hence

$$5s = s + 7 \quad \text{and} \quad s = \frac{7}{4}.$$

(5) Let (a_n) and (b_n) be monotone sequences. Prove or give a counterexample.

(a) The sequence (c_n) given by $c_n = a_n + b_n$ is monotone.

FALSE. Let $(a_n) = (1, 2, 2, 2, \dots)$ *and* $(b_n) = (2, 2, 1, 1, \dots)$. *Then* $(c_n) = (a_n + b_n) = (3, 4, 3, 3, \dots)$ *is not monotone.*

(b) The sequence (c_n) given by $c_n = a_n\, b_n$ is monotone.

FALSE. Let $(a_n) = (1, 2, 2, 2, \dots)$ *and* $(b_n) = (2, 2, 1, 1, \dots)$. *Then* $(c_n) = (a_n\, b_n) = (2, 4, 2, 2, \dots)$ *is not monotone.*