Math 3333 Name: SOLUTION

HW 6

Please, write clearly and justify all your statements using the material covered in class to get credit for your work.

(1) [3 Pts] Find an example of a sequence of real numbers satisfying each set of properties.

(a) Cauchy, but not monotone.

$$a_n = \frac{(-1)^n}{n}$$

(b) Monotone, but not Cauchy.

$$a_n = n$$

(c) Bounded, but not Cauchy.

$$a_n = (-1)^n$$

(2) [3 Pts] Prove or give a counterexample.

(a) Every bounded sequence has a Cauchy subsequence.

TRUE. By Bolzano-Weierstrass Lemma, every bounded sequence has a convergent subsequence. Since every convergent sequence is Cauchy, then every bounded sequence has a Cauchy subsequence.

(b) Every monotone sequence has a bounded subsequence.

FALSE. The sequence (n) *is monotone but has no bounded subsequence.*

(c) Every convergent sequence can be represented as the sum of two oscillating sequences.

TRUE. Let (s_n) *be a convergent sequence with* $\lim s_n = s$. *Next set* $a_n = \frac{s_n}{2} + (-1)^n$ *and* $b_n = \frac{s_n}{2} - (-1)^n$. *Then it is true that* $s_n = a_n + b_n$ *and that* (a_n) *and* (b_n) *are oscillating sequences. In fact* $\limsup a_n = \frac{s}{2} + 1$ *and* $\liminf a_n = \frac{s}{2} - 1$. *Similarly,* $\limsup b_n = \frac{s}{2} + 1$ *and* $\liminf b_n = \frac{s}{2} - 1$.

(3) [3 Pts] Let (s_n) and (t_n) be bounded sequences.

(a) Prove that $\limsup(s_n + t_n) \le \limsup s_n + \limsup t_n$.

(b) Find an example to show that equality may not hold in part (a).

(a) Proof. Since (s_n) *and* (t_n) *are bounded sequences, then there are* $s, t \in \mathbb{R}$ *such that* $s = \limsup s_n$ *and* $t = \limsup t_n$. *By definition, given* $\epsilon > 0$, *there exists* $N_1 = N_1(\epsilon) \in \mathbb{N}$ *such that* $s_n < s + \epsilon/2$ *if* $n > N_1$. *Similarly, there exists* $N_2 = N_2(\epsilon) \in \mathbb{N}$ *such that* $t_n < t + \epsilon/2$ *if* $n > N_2$. *Set* $N = \max(N_1, N_2)$. *It follows that, if* $n > N$ *then*

$$s_n + t_n < s + t + \epsilon.$$

Since ϵ *is arbitrary, we conclude that* $\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n$.

(b) Let $(s_n) = (1, 0, 1, 0, \dots)$ *and* $(t_n) = (0, 1, 0, 1, \dots)$. *Then*

$$1 = \limsup(s_n + t_n) < \limsup(s_n) + \limsup(t_n) = 2.$$

(4) [3 Pts] Show that each series is divergent.
(a) $\sum (-1)^n$

The series diverges since the sequence $a_n = (-1)^n$ *is not convergent to 0 (in fact it is not convergent).*

(b) $\sum \frac{n}{2n+1}$

The series diverges since the sequence $a_n = \frac{n}{2n+1}$ *is not convergent to 0 (in fact it converges to* $\frac{1}{2}$*).*

(c) $\sum \cos \frac{n\pi}{2}$

The series diverges since the sequence $a_n = \cos \frac{n\pi}{2}$ *is not convergent to 0. In fact* (a_n) *is not convergent as one can prove by noticing that the subsequences* (a_{2n+1}) *and* (a_{4m}) *have different limits, namely,* $\lim a_{2n+1} = 0$ *and* $\lim a_{4m} = 1$.

(5) [3 Pts] Find the sum of each series.
(a) $\sum_{n=1}^{\infty} \frac{1}{3^n}$

Using the formula for the geometric series, $\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1-1/3} = \frac{3}{2}$. *Hence* $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{3}{2} - 1 = \frac{1}{2}$.

(b) $\sum_{n=1}^{\infty} (-\frac{3}{4})^n$

Using the formula for the geometric series, $\sum_{n=0}^{\infty}(-\frac{3}{4})^n = \frac{1}{1+3/4} = \frac{4}{7}$*. Hence* $\sum_{n=1}^{\infty}(-\frac{3}{4})^n = \frac{4}{7} - 1 = -\frac{3}{7}$.

(c) $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$

Observe that

$$\sum_{n=2}^{N} \frac{1}{n(n-1)} = \sum_{n=2}^{N} \frac{1}{(n-1)} - \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots - \frac{1}{N-1} + \frac{1}{N-1} - \frac{1}{N}.$$

Hence $\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \lim_{N\to\infty} \sum_{n=2}^{N} \frac{1}{n(n-1)} = \lim_{N\to\infty}(1 - \frac{1}{N}) = 1$.

(6) [3 Pts] Let (a_n) *be a sequence of nonnegative real numbers. Prove that* $\sum a_n$ *converges iff the sequence of partial sums is bounded.*

Proof. *Since* $a_n \geq 0$ *for all* n*, then the sequence of partial sums* $(s_N) = (\sum_{n\leq N} a_n)$ *is monotone nondecreasing. It follows by the Monotone Convergence Theorem of sequences that* (s_N) *converges iff it is bounded.*

(7) [3 Pts] Determine whether or not the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ *converges. Justify your answer.*

Observe that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = \sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n+1-n} = \sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n}$$

The analysis of partial sums gives that

$$\begin{aligned} \sum_{n=1}^{N} \frac{1}{\sqrt{n+1}+\sqrt{n}} &= \sum_{n=1}^{N} \sqrt{n+1} - \sqrt{n} \\ &= \sqrt{2} - \sqrt{1} + \sqrt{3} - \sqrt{2} + \cdots + \sqrt{N+1} - \sqrt{N} \\ &= \sqrt{N+1} - 1. \end{aligned}$$

Hence $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = \lim_{N\to\infty} \sqrt{N+1} - 1 = \infty$*. This shows that the series diverges.*

(8) [3 Pts] Let (x_n) *be a sequence of real numbers and let* $y_n = x_n - x_{n+1}$ *for each* $n \in \mathbb{N}$.

(a) Prove that the series $\sum_{n=1}^{\infty} y_n$ *converges iff the sequence* (x_n) *converges.*

(b) If $\sum_{n=1}^{\infty} y_n$ converges, what is the sum?

Proof (a-b). A direct computation on the sequence of partial sums $(s_N) = (\sum_{n=1}^{N} y_n)$ shows that

$$\sum_{n=1}^{N} y_n = \sum_{n=1}^{N} (x_n - x_{n+1}) = x_1 - x_2 + x_2 - x_3 + \cdots - x_N + x_N - x_{N+1} = x_1 - x_{N+1}.$$

It follows from the last equation that $\sum_{n=1}^{\infty} y_n$ converges iff (x_n) converges.

If $\lim x_n = L$, it then follows from the calculation above that

$$\sum_{n=1}^{\infty} y_n = \lim_{N\to\infty} (x_1 - x_{N+1}) = x_1 - L.$$