Math 3333 Name: SOLUTION

<u>HW 8</u>

(1) Let $f : D \to \mathbb{R}$ be continuous at $c \in D$. Prove that there exists an $M > 0$ and a neighborhood U of c such that $|f(x)| \leq M$ for all $x \in U \cap D$.

Proof. *Since f is continuous at c, there exists a $\delta > 0$ such that if $|x-c| < \delta$ and $x \in D$, then $|f(x) - f(c)| < 1$. This implies that, for all $x \in D$ such that $|x - c| < \delta$, we have that $|f(x)| \leq 1 + |f(c)|$.*

(2) Determine the following limit

$$\lim_{x\to 0-} \frac{4x}{|x|}$$

(a) using the sequential definition;
(b) using the $\epsilon - \delta$ definition.

(a) Let x_n be a sequence converging to 0^-. That is, $\lim_n x_n = 0$ and, in addition, there is exists an $N > 0$ such that $x_n < 0$ if $n > N$. Then

$$\lim_{x\to 0-} \frac{4x}{|x|} = \lim_{n>N} \frac{4x_n}{|x_n|} = \lim_{n>N} \frac{4x_n}{(-x_n)} = 4.$$

(b) Given $\epsilon > 0$, let δ be any positive quantity. Then, if $-\delta < x < 0$, we have that

$$\left|\frac{4x}{|x|} + 4\right| = \left|\frac{4x}{(-x)} + 4\right| = |-4+4| = 0 < \epsilon.$$

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Show that f is not continuous at $x = 0$.
(b) Show that f has the intermediate property on any interval $[a, b] \in \mathbb{R}$, that is, if k is any value between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ such that $f(c) = k$.

(a) Let $x_n = \frac{1}{\pi/2+\pi n}$. *Then* $\lim_n x_n = 0$ *but*

$$\lim_n f(x_n) = \lim_n \sin(\tfrac{\pi}{2} + \pi n) = \begin{cases} 1 & \text{if } n \text{ even} \\ -1 & \text{if } n \text{ odd.} \end{cases}$$

Hence f *is not continuous at* 0.

(b) If $0 < a < b$ *or if* $a < b < 0$*, then* f *is continuous on* $[a, b]$ *and the intermediate value property holds. Let us consider the case where* $a < 0$ *and* $b > 0$*. WLOG assume that* $-1 \leq f(a) < k < f(b) \leq 1$ *(the endpoints are due to the range of the function* $\sin$*). By the Archimedean property, there is an* n *such that* $0 < \frac{1}{\pi/2+\pi n} < b$*. Clearly* $0 < \frac{1}{\pi/2+\pi(n+1)} < b$*. Note that* f *ranges continuously between -1 and 1 within the interval* $x \in [\frac{1}{\pi/2+\pi(n+1)}, \frac{1}{\pi/2+\pi n}]$*. Hence there is a* c *within this interval such that* $f(c) = k$*.*

(4) Show that any polynomial p of odd degree has at least one real root.

Proof. *Since* p *is a polynomial of odd degree, then either* $\lim_{x\to\infty} p(x) = \infty$ *and* $\lim_{x\to-\infty} p(x) = -\infty$ *or vice versa. In either case, this implies that* $p(x)$ *changes sign so that, by the Intermediate Value Theorem, there must be a point* x_0 *where* $p(x_0) = 0$*, that is, it must have one real root.*

(5) Let $f : [a, b] \to [a, b]$ be continuous. Prove that f must have a fixed point, that is, there is $c \in [a, b]$ such that $f(c) = c$. [Hint: Set $h(x) = f(x) - x$ and apply the Intermediate Value Theorem.]

Proof. *Set* $h(x) = f(x) - x$*. If* $f(a) = a$ *or* $f(b) = b$*, then the proof is complete. Let us consider the case where* $f(a) \neq a$ *and* $f(b) \neq b$*. Consider first the case where* $f(a) > a$ *and* $f(b) < b$*. It follows that* $h(a) > 0$ *and* $h(b) < 0$*. Hence,by the Intermediate Value Theorem, there must be a point* c *where* $h(c) = 0$ *and* $f(c) = c$*. The case* $f(a) < a$ *and* $f(b) > b$ *is similar. The cases where* $f(a) > a$ *and* $f(b) > b > a$ *or* $f(a) < a < b$ *and* $f(b) < b$ *are not possible since they violate the assumption that* $f : [a, b] \to [a, b]$ *continuously.*