Math 3333 Name: SOLUTION

HW 8

Please, write clearly and justify all your statements using the material covered in class to get credit for your work.

(1) Prove that the function $f(x) = \frac{1}{x}$ on $[2, \infty)$ is uniformly continuous by verifying the $\epsilon - \delta$ property.

Proof. *Observe that, for any $x, y \in [2, \infty)$,*

$$|\frac{1}{x} - \frac{1}{y}| = \frac{|y-x|}{xy} \le \frac{|y-x|}{4}.$$

Hence, given any $\epsilon > 0$, set $\delta = 4\epsilon$, then $|x-y| < \delta$ implies that $|\frac{1}{x} - \frac{1}{y}| < \epsilon$.

(2) Let $f : D \to \mathbb{R}$ be continuous at $c \in D$. Prove that there exists an $M > 0$ and a neighborhood U of c such that $|f(x)| \le M$ for all $x \in U \cap D$.

Proof. *Since f is continuous at c, there exists a $\delta > 0$ such that if $|x-c| < \delta$ and $x \in D$, then $|f(x) - f(c)| < 1$. This implies that, for all $x \in D$ such that $|x-c| < \delta$, we have that $|f(x)| \le 1 + |f(c)|$.*

(3) Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Proof. *Since $f(x) = \sqrt{x}$ is continuous on the close interval $[0, 2]$, it is also uniformly continuous on $[0, 2]$. It follows that, given any $\epsilon > 0$, there is a δ_1 such that $|x-y| < \delta_1$ implies $|\sqrt{x} - \sqrt{y}| < \epsilon$ for all $x, y \in [0, 2]$.*

If $x > 1$, observe that

$$|\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{\sqrt{x} + \sqrt{y}} \le \frac{|x-y|}{2}$$

Hence, given any $\epsilon > 0$, if we set $\delta_2 = 2\epsilon$, it follows that $|x-y| < \delta_2$ implies $|\sqrt{x} - \sqrt{y}| < \epsilon$ for all $x, y \in [1, \infty)$.

Set $\delta = \min\{\delta_1, \delta_2\}$. It then follows that $|x-y| < \delta$ implies that $|\sqrt{x} - \sqrt{y}| < \epsilon$ for all $x, y \in [0, \infty)$.

Note: For the proof to be complete, it is required for the intervals $[0, 2]$ and $[1, \infty)$ to have a nonempty overlap. If we only prove the result for $x \in [0, 1]$ and $x \in [1, \infty)$ then we cannot deal with the situation of points $|x-y|$ with $x \in [0, 1]$ and $y \in [2, \infty)$.

(4) Let f and g be two real-valued functions that are uniformly continuous on a set D. Prove that $f+g$ is uniformly continuous on D.

Proof. *By definition, given $\epsilon > 0$, there is a $\delta_1 > 0$ such that $|x-y| < \delta_1$ implies that $|f(x) - f(y)| < \epsilon/2$ for all $x, y \in D$, and there is a $\delta_2 > 0$ such that $x - y| < \delta_2$ implies that $|g(x) - g(y)| < \epsilon/2$ for all $x, y \in D$. Set $\delta = \min\{\delta_1, \delta_2\}$. It then follows that $|x-y| < \delta$ implies that*

$$|(f+g)(x) - (f+g)(y)| \leq |f(x) - f(y)| + |g(x) - g(y)| < \epsilon/2 + \epsilon/2 = \epsilon$$

$x, y \in D$.

(5) Find two real-valued functions f and g that are uniformly continuous on a set D, but such that their product $f\,g$ is not uniformly continuous on D.

Consider $f(x) = x$ and $g(x) = x$, $x \in \mathbb{R}$. Then f, g are uniformly continuous but $h(x) = f(x)\,g(x) = x^2$ is not uniformly continuous.

(6) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and periodic. Prove that f is bounded and uniformly continuous on $\mathbb{R}$.

Proof. *Let T be the interval of periodicity of f, that is $f(x) = f(x+T)$ for any x. Since f is continuous, then it is uniformly continuous on the close interval $[0, 2T]$. Hence, given any $\epsilon > 0$, there is a $\delta > 0$ such that $|x-y| < \delta$ implies that $|f(x) - f(y)| < \epsilon$, for any $x, y \in [0, 2T]$. Without loss of generality, we assume that $\delta < T$. Due to this assumption on δ, for any $x, y \in \mathbb{R}$ such that $|x-y| < \delta$, there is a $n \in \mathbb{Z}$ such that $x + nT, y + nT \in [0, 2T]$. Hence $|x-y| = |(x-nT) - (y-nT)| < \delta$ implies that $|f(x) - f(y)| = |f(x+nT) - f(y+nT)| < \epsilon$.*

(7) Determine the following limit

$$\lim_{x \to 0-} \frac{4x}{|x|}$$

(a) using the sequential definition;
(b) using the $\epsilon - \delta$ definition.

(a) Let x_n be a sequence converging to 0^-. That is, $\lim_n x_n = 0$ and, in addition, there is exists an $N > 0$ such that $x_n < 0$ if $n > N$. Then

$$\lim_{x \to 0-} \frac{4x}{|x|} = \lim_{n > N} \frac{4x_n}{|x_n|} = \lim_{n > N} \frac{4x_n}{(-x_n)} = 4.$$

(b) Given $\epsilon > 0$, let δ be any positive quantity. Then, if $-\delta < x < 0$, we have that

$$\left|\frac{4x}{|x|} + 4\right| = \left|\frac{4x}{(-x)} + 4\right| = |-4+4| = 0 < \epsilon.$$