HW 9

(1) Suppose that $f : (a, b) \to \mathbb{R}$ is continuous and that $f(r) = 0$ for every rational number $r \in (a, b)$. Prove that $f(x) = 0$ for all $x \in (a, b)$.

Proof. *Let $y \in (a, b)\backslash Q$. Arguing by contradiction, suppose $f(y) = \alpha \neq 0$. By the continuity of f, given any sequence $(x_n) \subset (a, b)$ converging to y, we have that $\lim f(x_n) = \alpha$. However, if we choose $(x_n) \subset (a, b) \cap Q$, we have that $\lim f(x_n) = 0$. This is a contradiction. Thus it must be $\alpha = 0$.*

(2) Let $f : D \to \mathbb{R}$ and $c \in D$. We say that f is *bounded on a neighborhood of* c if there exists a neighborhood U of c and a number M such that $|f(x)| \leq M$ for all $x \in U \cap D$

(a) Suppose that f is bounded on a neighborhood of each x in D and that D is compact. Prove that f is bounded on D.

(b) Suppose that f is bounded on a neighborhood of each x in D and that D is not compact. Show that f is not necessarily bounded on D, even when f is continuous.

(c) Suppose that $f : [a, b] \to \mathbb{R}$ has a limit at each x in $[a, b]$. Prove that f is bounded on $[a, b]$.

(a) **Proof.** *For every $x \in D$ there is a neighborhood U_x of x and a number M_x such that $|f(y)| \leq M_x$ for all $y \in U_x \cap D$. The set $\{U_x : x \in D\}$ is an open cover of D and D is compact. Hence there exists and open subcover of D; that is, we can find a finite set $x_1, \ldots, x_m \in D$ such that $D \subset \cup_{i=1}^m U_{x_i}$. Let $M = \max\{M_{x_i} : i = 1, \ldots, m\}$. It then follows that $|f(x)| \leq M$ for all $x \in D$.*

(b) *Let $f(x) = \frac{1}{x}$, $D = (0, \infty)$. f is bounded on a neighborhood of each x in D but f is not bounded.*

(c) **Proof.** *For every $x \in [a, b]$, we have that $\lim_{y \to x} f(y) = L_x$. By the properties of a limit, there is a deleted neighborhood U_x^* of x and a number M_x^* such that $|f(y)| \leq M_x^*$ for all $y \in U_x^*$. It then follows that $|f(y)| \leq M_x = \max\{f(x), M_x^*\}$ for all $y \in U_x$. Since $[a, b]$ is a compact set, it follows from part (a) above that we can find a finite set $x_1, \ldots, x_m \in [a, b]$ such that $|f(x)| \leq \max\{M_{x_1}, \ldots, M_{x_m}\}$ for all $x \in [a, b]$.*

(3) Prove that the function $f(x) = \frac{1}{x}$ on $[2, \infty)$ is uniformly continuous by verifying the $\epsilon - \delta$ property.

Proof. *Observe that, for any $x, y \in [2, \infty)$*

$$|\frac{1}{x} - \frac{1}{y}| = \frac{|y-x|}{xy} \leq \frac{|y-x|}{4}.$$

Hence, given any $\epsilon > 0$, set $\delta = 4\epsilon$, then $|x - y| < \delta$ implies that $|\frac{1}{x} - \frac{1}{y}| < \epsilon$.

(4) Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Proof. *Since $f(x) = \sqrt{x}$ is continuous on the close interval $[0, 2]$, it is also uniformly continuous on $[0, 2]$. It follows that, given any $\epsilon > 0$, there is a δ_1 such that $|x - y| < \delta_1$ implies $|\sqrt{x} - \sqrt{y}| < \epsilon$ for all $x, y \in [0, 2]$.*

If $x > 1$, observe that

$$|\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{\sqrt{x} + \sqrt{y}} \leq \frac{|x-y|}{2}$$

Hence, given any $\epsilon > 0$, if we set $\delta_2 = 2\epsilon$, it follows that $|x - y| < \delta_2$ implies $|\sqrt{x} - \sqrt{y}| < \epsilon$ for all $x, y \in [1, \infty)$.

Set $\delta = \min\{\delta_1, \delta_2\}$. It then follows that $|x-y| < \delta$ implies that $|\sqrt{x}-\sqrt{y}| < \epsilon$ for all $x, y \in [0, \infty)$.

Note: For the proof to be complete, it is required for the intervals $[0, 2]$ and $[1, \infty)$ to have a nonempty overlap. If we only prove the result for $x \in [0, 1]$ and $x \in [1, \infty)$ then we cannot deal with the situation of points $|x - y|$ with $x \in [0, 1]$ and $y \in [2, \infty)$.

(5) Let f and g be two real-valued functions that are uniformly continuous on a set D. Prove that $f + g$ is uniformly continuous on D.

Proof. *By definition, given $\epsilon > 0$, there is a $\delta_1 > 0$ such that $|x - y| < \delta_1$ implies that $|f(x) - f(y)| < \epsilon/2$ for all $x, y \in D$, and there is a $\delta_2 > 0$ such that $x - y| < \delta_2$ implies that $|g(x) - g(y)| < \epsilon/2$ for all $x, y \in D$. Set $\delta = \min\{\delta_1, \delta_2\}$. It then follows that $|x - y| < \delta$ implies that*

$$|(f + g)(x) - (f + g)(y)| \leq |f(x) - f(y)| + |g(x) - g(y)| < \epsilon/2 + \epsilon/2 = \epsilon$$

$x, y \in D$.

(6) Find two real-valued functions f and g that are uniformly continuous on a set D, but such that their product $f\,g$ is not uniformly continuous on D.

Consider $f(x) = x$ *and* $g(x) = x$, $x \in \mathbb{R}$. *Then* f, g *are uniformly continuous but* $h(x) = f(x)\,g(x) = x^2$ *is not uniformly continuous.*