Math 3333 Name: SOLUTION

Quiz/HW 4

Please, write clearly and justify all your statements using the material covered in class to get credit for your work.

(1) [4 Pts] Mark each statement as True or False. If False, show a counter-example. If True, justify your answer.

(a) The set $S = \{\frac{1}{n} : n \in \mathbb{N}\}$ is compact.

False. The set S is not closed since 0 is an accumulation point of S but $0 \notin S$.

(b) If $S \subset \mathbb{R}$ is compact and x is an accumulation point of S, then $x \in S$.

True. If a set is compact then it is closed and it must contain all its accumulation points.

(c) If $S \subset \mathbb{R}$ is a compact, then there is at least one point in $\mathbb{R}$ that is an accumulation point of S.

False. The set $S = \{1, 2, 3\}$ is compact since closed and bounded but it contains no accumulation points.

(d) If a set S has a maximum and a minimum, then S is a compact set.

False. The set $[0, 1) \cup (2, 3]$ has minimum ($x = 0$) and maximum ($x = 3$) but is not a closed set, hence not a compact one.

(2) [4 Pts] Let (s_n) be a sequence such that $\lim_{n\to\infty} s_n = 0$ and (t_n) be a bounded sequence. Prove that the sequence $(s_n\, t_n)$ is convergent.

I will prove that $\lim_{n\to\infty} s_n\, t_n = 0$.
Since (t_n) is bounded, there is an $M > 0$ such that $t_n < M$ for all $n \in \mathbb{N}$.
Since $\lim_{n\to\infty} s_n = 0$, given any $\epsilon > 0$, there exists and $N = N(\epsilon)$ such that $|s_n| < \frac{\epsilon}{M}$ if $n > N$.
It follows that, given $\epsilon > 0$, there exists $N = N(\epsilon)$ such that $|s_n\, t_n| < \frac{\epsilon}{M} M = \epsilon$ if $n > N$.
This shows that $\lim_{n\to\infty} s_n\, t_n = 0$.

(3)[2 Pts] Prove or give a counterexamples:

(a) If (s_n) and (t_n) are divergent sequences, then $(s_n + t_n)$ diverges.

FALSE. Let $(s_n) = (n)$ and $(t_n) = (-n)$. $(s_n + t_n) = 0$ convergent.

(b) If (s_n) is convergent and (t_n) is bounded, then $(s_n\, t_n)$ converges.

FALSE. Let $(s_n) = 1$ and $(t_n) = (-1)^n$. $(s_n\, t_n) = 1$ divergent.