Test #1

This is a closed-book, no-notes test. Please, write clearly and justify your arguments using the material covered in class to get credit for your work.

(1)[4 Pts] Use an argument by induction to prove that, if $x \geq 0$, then

$$(1+x)^n \geq 1 + n\,x \quad \text{for all } n \in \mathbb{N}.$$

Proof.
- For $n = 1$, $(1+x)^1 \geq 1 + 1 \cdot x = 1 + x$.
- Assume $(1+x)^n \geq 1 + n\,x$ for some $n \in \mathbb{N}$.
- We now derive the case $n+1$. Using the statement above for n, we observe that

$$(1+x)^{n+1} = (1+x)(1+x)^n \geq (1+x)(1+nx) = 1+x+nx+nx^2 \geq 1+x+nx = 1+(n+1)x.$$

This shows that statement is true for $n+1$, hence it is true for all $n \in \mathbb{N}$. □

(2)[6 Pts] Let A, B be subsets of $\mathbb{R}$.

(a) State the definition of *boundary* for a subset S of $\mathbb{R}$.
(b) Prove that $\text{bd}(A \cup B) \subset \text{bd}(A) \cup \text{bd}(B)$.
(c) Show that the converse containment $\text{bd}(A) \cup \text{bd}(B) \subset \text{bd}(A \cup B)$ may fail by giving a counterexample, i.e., find sets A, B such that $\text{bd}(A) \cup \text{bd}(B) \not\subset \text{bd}(A \cup B)$.

(a) The boundary of S is the set of its boundary points. A boundary point of S is a point x such that for any $\epsilon > 0$ we have that $N(x,\epsilon) \cap S \neq \emptyset$ and $N(x,\epsilon) \cap S^c \neq \emptyset$.

(b) *Proof.* We will show that, for any $x \in \text{bd}(A \cup B)$ then $x \in \text{bd}(A)$ or $x \in \text{bd}(B)$.

If $x \in \text{bd}(A \cup B)$, then for any $\epsilon > 0$ we have that $N(x,\epsilon) \cap (A \cup B) \neq \emptyset$ and $N(x,\epsilon) \cap (A \cup B)^c \neq \emptyset$. Note that $(A \cup B)^c = (A^c \cap B^c)$. Hence, for any $\epsilon > 0$ we have that $N(x,\epsilon) \cap A \neq \emptyset$ and $N(x,\epsilon) \cap A^c \neq \emptyset$ or $N(x,\epsilon) \cap B \neq \emptyset$ and $N(x,\epsilon) \cap B^c \neq \emptyset$. That is, we have that $x \in \text{bd}(A)$ or $x \in \text{bd}(B)$. □

(c) Let $A = [1,3]$ and $B = (2,4)$. Observe that $\text{bd}(A) = \{1,3\}$, $\text{bd}(B) = \{2,4\}$ and $\text{bd}(A \cup B) = \text{bd}([1,4)) = \{1,4\}$. Hence in this case $\text{bd}(A \cup B)$ is strictly contained in $\text{bd}(A) \cup \text{bd}(B)$ so that $\text{bd}(A) \cup \text{bd}(B) \not\subset \text{bd}(A \cup B)$

(3)[5 Pts] Find a set (or sets) satisfying the description below, or explain why they do not exist.

(a) A set $S \in \mathbb{R}$ that is neither open nor closed.

$S = [1, 2)$

(b) A set $S \in \mathbb{R}$ that has a maximum, a minimum and is not closed.

$S = [1, 2) \cap (3, 4]$. Note that $\min S = 1$ and $\max S = 4$ but S is not closed.

(c) A collection of open sets A_n such that $\cap_n A_n$ is not open.

$A_n = (-\frac{1}{n}, \frac{1}{n})$. Then $\cap_n A_n = \{0\}$, closed set.

(d) A collection of open sets A_n such that $\cup_n A_n$ is not open.

Not possible. By theorem in class, the union of any collection of open sets is open.

(e) An unbounded set containing no accumulation points.

The set $\mathbb{N}$ is unbounded and contains no accumulation points.