Math 3333 – SPRING 2025 Name: SOLUTION

Test #1

This is a closed-book, no-notes test. Please, write clearly and justify your arguments using the material covered in class to get credit for your work.

(1)[4 Pts] Let $x \neq 1$. Use an argument by induction to prove that:

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x} \quad \text{for all } n \geq 0.$$

Proof.
- For $n = 0$, $\sum_{i=0}^{0} x^i = 1 = \frac{1-x^1}{1-x}$
- Assume $\sum_{i=0}^{k} x^i = \frac{1-x^{k+1}}{1-x}$ for some $k \in \mathbb{N}$.
- We now derive the case $k + 1$. Using the statement above for k, we observe that

$$\sum_{i=0}^{k+1} x^i = \sum_{i=0}^{k} x^i + x^{k+1} = \frac{1 - x^{k+1}}{1 - x} + x^{k+1} = \frac{1 - x^{k+2}}{1 - x}$$

This shows that statement is true for $k + 1$, hence it is true for all $n \in \mathbb{N}$. □

(2)[3 Pts] Find a set satisfying the description below, or explain why it does not exist.

(a) A set $S \in \mathbb{R}$ that is neither open nor closed.
$S = [1, 2)$

(b) A set $S \in \mathbb{R}$ that has a maximum, a minimum and is not closed.
$S = [1, 2) \cap (3, 4]$. Note that $\min S = 1$ and $\max S = 4$ but S is not closed.

(c) An unbounded set containing no accumulation points.
The set $\mathbb{N}$ is unbounded and contains no accumulation points.

(3)[5 Pts] Let A, B be nonempty subsets of $\mathbb{R}$.

(a) State the definition of *boundary* for a subset S of $\mathbb{R}$.

(b) Prove that $\text{bd}(A \cup B) \subset \text{bd}(A) \cup \text{bd}(B)$.

(c) Show that the converse containment $\text{bd}(A) \cup \text{bd}(B) \subset \text{bd}(A \cup B)$ may fail by giving a counterexample, i.e., find sets A, B such that $\text{bd}(A) \cup \text{bd}(B) \not\subset \text{bd}(A \cup B)$.

(a) The boundary of S is the set of its boundary points. A boundary point of S is a point x such that for any $\epsilon > 0$ we have that $N(x, \epsilon) \cap S \neq \emptyset$ and $N(x, \epsilon) \cap S^c \neq \emptyset$.

(b) *Proof.* We will show that, for any $x \in \text{bd}(A \cup B)$ then $x \in \text{bd}(A)$ or $x \in \text{bd}(B)$.

If $x \in \text{bd}(A \cup B)$, then for any $\epsilon > 0$ we have that $N(x, \epsilon) \cap (A \cup B) \neq \emptyset$ and $N(x, \epsilon) \cap (A \cup B)^c \neq \emptyset$. Note that $(A \cup B)^c = (A^c \cap B^c)$. Hence, for any $\epsilon > 0$ we have that $N(x, \epsilon) \cap A \neq \emptyset$ and $N(x, \epsilon) \cap A^c \neq \emptyset$ or $N(x, \epsilon) \cap B \neq \emptyset$ and $N(x, \epsilon) \cap B^c \neq \emptyset$. That is, we have that $x \in \text{bd}(A)$ or $x \in \text{bd}(B)$. □

(c) Let $A = [1, 3]$ and $B = (2, 4)$. Observe that $\text{bd}(A) = \{1, 3\}$, $\text{bd}(B) = \{2, 4\}$ and $\text{bd}(A \cup B) = \text{bd}([1, 4)) = \{1, 4\}$. Hence in this case $\text{bd}(A \cup B)$ is strictly contained in $\text{bd}(A) \cup \text{bd}(B)$ so that $\text{bd}(A) \cup \text{bd}(B) \not\subset \text{bd}(A \cup B)$