Math 3333 – Spring 2022 Name: SOLUTION

Test #2

This is a closed-book, no-notes test.

[1] [5 Pts]

(a) State the definition of *convergence* for a sequence of real numbers.

A sequence of real numbers (a_n) *converges to* $L \in \mathbb{R}$ *if, given any* $\epsilon > 0$, *there exists an* $N \in \mathbb{N}$ *(where* N *may depend on* ϵ*) such that* $|a_n - L| < \epsilon$ *whenever* $n > N$.

(b) Use the definition of convergence to prove that if (a_n) converges to L, then $(|a_n|)$ converges to $|L|$.

Proof. *Assume that* (a_n) *converges to* L. *Hence, by definition, given any* $\epsilon > 0$, *there is an* $N = N(\epsilon)$ *such that*

$$|a_n - L| < \epsilon \quad \text{if } n > N.$$

It follows that

$$||a_n| - |L|| = ||a_n - L + L| - |L|| \leq |a_n - L| < \epsilon \quad \text{if } n > N.$$

This shows that $(|a_n|)$ *converges to* $|L|$.

[2] [5 Pts] Consider the sequence of real numbers defined by

$$s_1 = 2 \text{ and } s_{n+1} = \sqrt{2s_n + 3} \text{ for } n \in \mathbb{N}.$$

(a) Prove that (s_n) is convergent

(b) Find the limit of (s_n).

Claim: $|s_n| \leq 3$ *for all* $n \in \mathbb{N}$.
Proof by induction:

(1) $|s_1| = 2 \leq 3$

(2) Assume $|s_k| \leq 3$ *for some* $k \in \mathbb{N}$

(3) Using step (2), we have: $|s_{k+1}| = \sqrt{2s_k + 3} \leq \sqrt{6 + 3} = 3$.

Claim: $s_{n+1} \geq s_n$ *for all* $n \in \mathbb{N}$.
Proof by induction:

(1) $s_2 = \sqrt{7} > 2 = s_1$

(2) *Assume* $s_{k+1} \geq s_k$ *for some* $k \in \mathbb{N}$.

(3) *Using step (2), we have:* $s_{k+2} = \sqrt{2s_{k+1} + 3} \geq \sqrt{2s_k + 3} = s_{k+1}$.

By the Monotone bounded theorem for sequences, it follows that (s_n) *converges. Let* $\lim s_n = L$. *Then*

$$L = \lim s_{n+1} = \lim \sqrt{2s_n + 3} = \sqrt{2L + 3}$$

Hence $L^2 - 2L - 3 = 0$ *and* $L = 3$ *or* $L = -1$. *Since the limit of the sequence is nonnegative, it must be* $L = 3$.

(3)[5 Pts] For each of the following statements, either prove it (you can use theorems discussed in class) or give a counterexample.

(a) Every bounded non-negative sequence of real numbers converges.

FALSE. $(a_n) = (1+(-1)^n)$ *is bounded and non-negative but not convergent.*

(b) If $(|s_n|)$ is a convergent sequence of real numbers, then (s_n) is also convergent.

FALSE. $(s_n) = (-1^n)$ *is not convergent but* $(|s_n|) = (1^n)$ *is convergent.*

(c) If the sequence of real numbers (s_n) diverges to $+\infty$ and the sequence of real numbers (t_n) is both bounded and positive ($t_n > 0$ for all n), then $(t_n s_n)$ diverges to $+\infty$.

FALSE. Let $(s_n) = (n)$ *and* $(t_n) = (\frac{1}{n})$*; then* $s_n t_n = 1$ *and this sequence is convergent.*

(d) If the sequence of real numbers (s_n) diverges to $+\infty$ and the sequence of real numbers (t_n) has apositive limit $\lim t_n = L > 0$, then $(t_n s_n)$ diverges to $+\infty$.

TRUE. By problem (3) in homework 5, whose proof is available online and was presented in class.

(e) If $\lim s_n = 0$ and the sequence of real numbers (t_n) is non-negative, that is $t_n \geq 0$ for all n, then $(t_n s_n)$ converges.

FALSE. Let $(s_n) = (1/n)$ *and* $(t_n) = (n^2)$*; then* $\lim s_n t_n = \infty$.