Math 3333 – Spring 2015 Name:

Test #2

[Please, write your solution legibly. You must justify your answers to get credit for your work.

(1) [4 Pts] Suppose that $\lim s_n = 1$. Use the definition of limit to prove that there exists an $N \in \mathbb{N}$ such that $s_n < 1.1$ for all $n \geq N$.

Since (s_n) converges, there exists an $N \in \mathbb{N}$ such that $|s_n - 1| < 0.1$ if $n \geq N$ (we apply the definition of convergence with $\epsilon = 0.1$). This implies that $0.9 < s_n < 1.1$ if $n \geq N$.

(2) Prove that if $\lim_{n\to\infty} s_n = -\infty$ and if (t_n) is a bounded sequence, then

$$\lim_{n\to\infty} (s_n + t_n) = -\infty$$

Proof. Since (t_n) is bounded, there exists an N_1 and an $L > 0$ such that $|t_n| < L$ if $n > N_1$. That is, $-L < t_n < L$ for all $n > N_1$

Since $\lim_{n\to\infty} s_n = -\infty$, given any $M > 0$, there exists an N_2 such that $s_n < -M - L$ if $n > N_2$.

Hence, provided $n > \max\{N_1, N_2\}$, we have that $s_n + t_n < -M$. Since M is arbitrary, this proves that $\lim_{n\to\infty}(s_n + t_n) = -\infty$.

(3)[5 Pts] Consider the sequence of real numbers defined by

$$s_1 = 3 \text{ and } s_{n+1} = \frac{1}{2}\sqrt{s_n + 3} \text{ for } n \in \mathbb{N}.$$

(a) Prove that (s_n) is convergent.

(b) Find the limit of (s_n).

Claim: $s_n \geq 0$ for all $n \in \mathbb{N}$.
Proof by induction:

(1) $s_1 = 3 \geq 0$

(2) Assume $s_n \geq 0$

(3) $s_{n+1} = \frac{1}{2}\sqrt{s_n + 3} \geq \frac{1}{2}\sqrt{0 + 3} = \frac{\sqrt{3}}{2} \geq 0.$

Claim: $s_{n+1} \leq s_n$ for all $n \in \mathbb{N}$.
Proof by induction:

(1) $s_2 = \frac{1}{2}\sqrt{6} < 3 = s_1$

(2) Assume $s_{n+1} \leq s_n$.

(3) $s_{n+2} = \frac{1}{2}\sqrt{s_{n+1}+3} \leq \frac{1}{2}\sqrt{s_n+3} = s_{n+1}$.

By the Monotone bounded theorem for sequences, it follows that (s_n) *converges. Let* $\lim s_n = L$. *Then*

$$2L = 2\lim s_{n+1} = \lim \sqrt{s_n+3} = \sqrt{L+3}$$

Hence $4L^2 - L - 3 = 0$ *with roots* $L = 1$ *or* $L = -3/4$. *Since the limit of the sequence is nonnegative, it must be* $L = 1$.

(4)[6 Pts] For each of the following statements, state whether it True or False. If True, prove it; if False, give a counterexample.

(a) If $(|s_n|)$ is a convergent sequence, then (s_n) is also convergent.

FALSE. Let $s_n = (-1)^n$. $(|s_n|) = (1, 1, 1, \dots)$ *is a convergent sequences but* (s_n) *is not convergent.*

(b) If the sequence of real numbers (s_n) diverges to $+\infty$, the sequence of real numbers (t_n) is bounded and $t_n \geq 0$ for all n, then $(s_n t_n)$ diverges to $+\infty$.

FALSE. Let $s_n = n$ *and* $t_n = \frac{1}{n}$. *We have that* $\lim s_n = \infty$, (t_n) *bounded and non-negative. However the product is not diverging since* $\lim s_n t_n = 0$.

(c) If the sequence of real numbers (s_n) converges and the sequence of real numbers (t_n) is bounded, then $(t_n s_n)$ also converges.

FALSE. Let $(s_n) = (1, 1, 1, \dots)$ *and* $t_n = (-1)^n$. *Then* $(s_n t_n) = ((-1)^n)$ *which is divergent.*

(d) If $S \subset \mathbb{R}$ is a compact nonempty set, then there is at least one point in $\mathbb{R}$ that is an accumulation point of S.

FALSE. The set $S = \{1, 2, 3\}$ *is compact since closed and bounded but it contains no accumulation points.*

(e) The set $S = \{\frac{1}{n} : n \in \mathbb{N}\}$ is a compact set.

FALSE. The set S *is not closed since the point* 0 *is an accumulation point of* S *but does not belong to* S. *Hence* S *is not compact.*