Math 3333 – Spring 2022 Name: SOLUTION

Test #3

Please, read carefully the statement of the problem. You must justify your arguments using the methods and results presented in class.

(1)[5 Pts]

(a) State the $\epsilon - \delta$ definition of continuity for a function $f : D \mapsto \mathbb{R}$, where $D \subset \mathbb{R}$.

(b) Prove that the function $f(x) = x^2$ on $(0, 7)$ is continuous by verifying the $\epsilon - \delta$ property and show that you can choose a δ independent of x.

HINT: note that f is defined on $(0, 7)$, not on all of $\mathbb{R}$.

(a) Let $f : D \mapsto \mathbb{R}$. f is continuous at $x_0 \in D$ if, given $\epsilon > 0$, there is a $\delta > 0$ such that $|x - x_0| < \delta$ and $x \in D$ imply that $|f(x) - f(x_0)| < \epsilon$. f is continuous on D if f is continuous for every $x \in D$.

(b) We have that $|f(x) - f(y)| = |x^2 - y^2| = |x - y||x + y| \le 14|x - y|$ for $x, y \in (0, 7)$.

Hence, given any $\epsilon > 0$, let $\delta = \epsilon/14$. It follows that $|f(x) - f(y)| < \epsilon$ if $|x - y| < \delta$, for all $x, y \in (0, 7)$.

(2)[5 Pts] Let

$$f(x) = \begin{cases} 1 + \sqrt{x} & \text{if } 0 \le x < 1; \\ a - x & \text{if } x \ge 1. \end{cases}$$

(a) Use the definition of one-sided limits to determine the limit $\lim_{x \to 1^-} f(x)$.

(b) Find the value of a so that f continuous at $x = 1$. Justify your answer by proving that f is continuous at a once you assign the correct value.

(a) Observe that, for $0 \le x < 1$, $|f(x) - 2| = |\sqrt{x} - 1| = \frac{|x-1|}{|\sqrt{x}+1|} \le |x - 1|$. Hence, given any $\epsilon > 0$, let $\delta = \epsilon$. Then $|f(x) - 2| < \epsilon$ if $1 - \delta \le x < 1$. This shows that $\lim_{x \to 1^-} f(x) = 2$.

(b) Let $a = 3$, then $f(1) = 2$. Claim: $\lim_{x \to 1^+} f(x) = 2$. In fact, given any $\epsilon > 0$, let $\delta = \epsilon$. Then $|f(x) - 2| = |x - 1| < \epsilon$ if $1 < x < 1 + \delta$.

Since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 2 = f(1)$, then f is continuous at $x = 1$.

(3)[5 Pts] Let

$$g(x) = \begin{cases} 3x & \text{if } x \text{ is rational;} \\ 2x + 1 & \text{if } x \text{ is irrational.} \end{cases}$$

(a) Prove that g is discontinuous at $x = \frac{1}{2}$.

(b) Is g continuous at $x = 1$? Prove it, if you think it is; otherwise, prove that it is not.

(a) Let (x_n) be a sequence of irrationals converging to $\frac{1}{2}$. Then $\lim_n g(x_n) = \lim_n 2x_n + 1 = 2$. However $g(\frac{1}{2}) = \frac{3}{2}$. Hence g is not continuous at $x = \frac{1}{2}$.

(b) $g(1) = 3$. Given any $\epsilon > 0$, let $\delta = \epsilon/3$. Then $|g(x) - 3| \le 3|x - 1| < \epsilon$ if $|x - 1| < \delta$.

Note that the inequality $|g(x) - 3| \le 3|x - 1|$ follows from observation that $|g(x) - 3| = 3|x - 1|$ if x is rational and $|g(x) - 3| = 2|x - 1|$ if x is irrational.