**Math 3333 – Spring 2025** Name:

# <u>Test #3</u>

This is a closed-book, no-notes test. Please, write clearly and justify your arguments using the material covered in class to get credit for your work.

(1)[6 Pts]

(a) Let

$$f(x) = x^2 \sin \frac{1}{x}.$$

Prove that $f$ is continuous at $x = 0$ or show that it is not.

(b) Let

$$g(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that $g$ is continuous at $x = 0$ or show that it is not.

(c) Let

$$h(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0; \\ 1 & \text{if } x = 0. \end{cases}$$

Prove that $h$ is continuous at $x = 0$ or show that it is not.

*(a) $f(x)$ is not continuous at $x = 0$ since it not defined at $x = 0$.*

*(b) To show that $g$ is continuous at $x = 0$, it suffices to show that $\lim_{x \to 0} g(x) = g(0) = 0$. For that, let $(x_n)$ be any sequence such that $\lim x_n = 0$. Notice that, for any $x_n \neq 0$, we have that $|g(x_n)| = x_n^2 |\sin \frac{1}{x_n}| \leq x_n^2$. Hence $\lim |g(x_n)| = 0$ and this implies that $\lim g(x_n) = 0$.*

*(c) By part (b) $\lim_{x \to 0} h(x) = 0$. However, $h(0) = 1 \neq \lim_{x \to 0} h(x)$. This implies that $h$ is not continuous at $x = 0$.*

(2)[5 Pts] Consider the function

$$f(x) = x^2, \quad x \in D = [0, 3]$$

(a) Use the $\epsilon - \delta$ characterization of continuity to prove that $f$ is continuous at $x_0$ in $D$.

(b) Show that $\delta$ in the argument above can be chosen independently of $x \in D$, that is, $\delta$ depends on $\epsilon$ but not on $x$.

(b) Determine a bound $\delta$ independent of $x$ such that $|x - 2| < \delta$ implies

$$|x^2 - 4| < 0.01.$$

*(a) We need to show that, given $\epsilon > 0$, for $x, x_0 \in D$, there is a $\delta$ such that $|x - x_0| < \delta$ implies $|x^2 - x_0^2| < \epsilon$.*

*Write $|x^2 - x_0^2| = |x - x_0||x + x_0|$. If $|x - x_0| < 1$, then $|x| < |x_0| + 1$ and $|x^2 - x_0^2| = |x + x_0||x - x_0| < (2|x_0| + 1)|x - x_0| < 7|x - x_0|$, where we used the fact that $|x_0| \leq 3$ due to the assumption on $D$.*

*Hence, to have $|x^2 - x_0^2| < \epsilon$, it is sufficient to require that $7|x - x_0| < \epsilon$ and $|x - x_0| < 1$. Let $\delta = \min(1, \epsilon/7)$. Then, if $|x - x_0| < \delta$ it follows that $|x^2 - x_0^2| < \epsilon$.*

*(b) The argument above shows that $\delta$ can be chosen independently of $x$ since $|x| < 3$.*

*(c) Using the conclusion in part (a) with $x_0 = 2$, it follows that we can choose $\delta < \frac{1}{7}0.01$; that is, $|x - 2| < \frac{1}{7} \cdot 0.01$ implies that $|x^2 - 4| < 0.01$.*

(3)[5 Pts] For each of the following statements, prove it (you can use the theorems presented in class) or give a counterexample.

(a) If a sequence $(s_n)$ is Cauchy, then it is a bounded sequence.

TRUE: Any Cauchy sequence is convergent hence it is also bounded.

(b) If a sequence $(s_n)$ is bounded, then it is a Cauchy sequence.

FALSE: $s_n = (-1)^n$ is bounded but not Cauchy

(c) Every monotone sequence $(s_n)$ has a convergent subsequence.

FALSE: $s_n = n$ is monotone with no convergent subsequence.

(d) If $\lim s_n = 0$, then $\sum s_n$ is a convergent series.

FALSE: Take $s_n = \frac{1}{n}$. Then $\lim \frac{1}{n} = 0$ but $\sum \frac{1}{n}$ diverges

(e) If $f : D \to \mathbb{R}$ is continuous then $f$ is bounded on $D$.

FALSE: Consider $f(x) = \frac{1}{x}$, for $x > 0$. $f$ is continuous on $D$ but $f$ is unbounded on $D$ since $\lim_{x \to 0} f(x) = \infty$.