Math 3339 – Spring 2023 Name: SOLUTION

Test #1

Please, write legibly and justify all your steps to get credit for your work.

(1) [5 Pts] *Suppose that* $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ *are the possible outcomes of a random experiment and suppose that each number is equally likely. Consider the event:* A_1 = *the number is even or 0,* A_2 = *the number is between 4 and 7 inclusive.*

(i) Are A_1 *and* A_2 *disjoint events? Justify using the definition.*

(ii) Compute $P(A_1 \cap A_2)$, $P(A_1 \cup A_2)$.

(iii) Are A_1 *and* A_2 *independent events? Justify using the definition.*

Note that $A_1 = \{0, 2, 4, 6, 8\}$, $A_2 = \{4, 5, 6, 7\}$, $A_1 \cap A_2 = \{4, 6\}$, $A_1 \cup A_2 = \{0, 2, 4, 5, 6, 7, 8\}$

(i) Since the intersection $A_1 \cap A_2 = \{4, 6\}$ is non-empty, the events are NOT DISJOINT.

(ii) $P(A_1 \cap A_2) = \frac{2}{10} = \boxed{\frac{1}{5}}$, $P(A_1 \cup A_2) = \frac{7}{10} = \boxed{\frac{4}{5}}$.

(iii) $P(A_1) = \frac{5}{10} = \boxed{\frac{1}{2}}$, $P(A_2) = \frac{4}{10} = \boxed{\frac{2}{5}}$,

$P(A_1)P(A_2) = \frac{1}{2} \cdot \frac{2}{5} = \frac{1}{5}$. Same as $P(A_1 \cap A_2) = \frac{1}{5}$. Hence events are INDEPENDENT.

(2)[4 Pts] *If* $P(A) = 0.3$ *and* $P(B) = 0.5$ *and* $P(A \cup B) = 0.6$, *find:*

(i) $P(A^c \cup B^c)$;

(ii) $P(A^c \cap B)$.

You need to justify your solution analytically.

(i) $P(A^c \cup B^c) = P((A \cap B)^c) = 1 - P(A \cap B) = \boxed{0.8}$.

(ii) $P(A^c \cap B) = P(B) - P(A \cap B) = \boxed{0.3}$.

(3)[4 Pts] Let the probability mass function $f(x)$ of a discrete random variable X be

x	0	1	2	3
$f(x)$	0.4	0.3	0.2	0.1

(a) Sketch the cumulative distribution function $F(x)$.

(b) Calculate the mean μ and the variance σ^2 of Z.

(a)

(b) $\mu = E(X) = \sum_{x=0}^{3} x\, f(x) = 0 \cdot 0.4 + 1 \cdot 0.3 + 2 \cdot 0.2 + 3 \cdot 0.1 = 1$

$E(X^2) = \sum_{x=0}^{3} x^2\, f(x) = 0 \cdot 0.4 + 1 \cdot 0.3 + 4 \cdot 0.2 + 9 \cdot 0.1 = 2.0.$

$\sigma^2 = E(X^2) - \mu^2 = 2.0 - (1.0)^2 = 1.0$

(4)[8 Pts] *Five cards are drawn successively at random and without replacement from an ordinary 52-deck of playing cards. Compute the following probabilities and write your solutions in terms of binomial coefficients or other fractions.*

(i) Compute the probability that 2 cards are spade, 2 cards are club and 1 card is neither spade nor club.

(ii) Compute the probability that at least one card is a spade.

(iii) Compute the probability that the sequence "spade, club, club, spade, spade" is observed in that particular order.

(iv) Compute the probability that the third ace appears in the fifth draw.

(i) $P(\text{combination of } 2\spadesuit, 2\clubsuit, else) = \boxed{\frac{\binom{13}{2}\binom{13}{2}\binom{26}{1}}{\binom{52}{5}}}$

$=$ `choose(13,2)*choose(13,2)*26/choose(52,5)` $= 0.06086435$

(ii) $P(\text{at least } 1\spadesuit) = 1 - P(\text{no } \spadesuit) = \boxed{1 - \frac{\binom{39}{5}}{\binom{52}{5}}}$

$=$ `1-choose(39,5)/choose(52,5)` $= 0.7784664$

(iii) $P(\spadesuit, \clubsuit, \clubsuit, \spadesuit, \spadesuit) = \boxed{\frac{13}{52}\,\frac{13}{51}\,\frac{12}{50}\,\frac{12}{49}\,\frac{11}{48}}$

$=$ `(13*13*12*12*11)/(52*51*50*49*48)` $= 0.0008583433$

(iv) $P(2A \text{ in 4 draws})P(A \text{ on fifth draw}|(2A \text{ in 4 draws})) = \boxed{\frac{\binom{4}{2}\binom{48}{2}}{\binom{52}{4}}\,\frac{2}{48}}$

$=$ `choose(4,2)*choose(48,2)*2/(48*choose(52,4))` $= 0.001041647$

(5)[4 Pts] *A biological laboratory is interested in determining whether a certain trace of impurity is present in a product. The (prior) probability of the impurity being present is 40%. An experiment has a probability 95 % of detecting the impurity when it is present. The probability of not detecting the impurity when it is absent is 90 %. Calculate the posterior probability that the impurity is present, that is, that the probability that the impurity is present when the experiment signals the detection of the impurity.*

NOTE: Please, define the relevant events and write the formula you are applying before computing the numerical solution with 3 decimal digits.

Denote: I=impurity present in product, D= experiment detects impurity.
Data: $P(I) = 0.4$, $P(D|I) = 0.95$, $P(D^c|I^c) = 0.9$.
Derived data: $P(I^c) = 0.6$, $P(D|I^c) = 0.1$.
Bayes' Theorem:

$$P(I|D) = \frac{P(D|I)P(I)}{P(D|I)P(I) + P(D|I^c)P(I^c)} = \boxed{\frac{(0.95)\,(0.4)}{(0.95)\,(0.4) + (0.1)\,(0.6)}} = 0.864$$