10.3.1 (a) $\mathfrak{F}[f](w) = \frac{1}{2\pi}\int_{\mathbb{R}} f(x)e^{iwx}\,dx$

By the linearity of INTEGRATION

$$\mathfrak{F}[c_1 f + c_2 g](w) = \frac{1}{2\pi}\int (c_1 f(x) + c_2 g(x))\, e^{iwx}\,dx =$$

$$= \frac{c_1}{2\pi}\int f(x)e^{iwx}\,dx + \frac{c_2}{2\pi}\int g(x)e^{iwx}\,dx$$

$$= c_1\,\mathfrak{F}[f](w) + c_2\,\mathfrak{F}[g](w)$$

10.3.3 f real $\Rightarrow f = \overline{f}$ $\qquad \overline{f}$ = complex conjugate of f

$$\overline{\mathfrak{F}[f](w)} = \overline{\frac{1}{2\pi}\int f(x)e^{iwx}\,dx} = \frac{1}{2\pi}\int f(x)e^{-iwx}\,dx$$

$$= \mathfrak{F}[f](-w)$$

10.3.5 let $f_\beta(x) = f(x-\beta)$

$$\mathfrak{F}[f_\beta](w) = \frac{1}{2\pi}\int f(x-\beta)e^{iwx}\,dx = \frac{1}{2\pi}\int f(x-\beta)e^{iw(x-\beta)}e^{iw\beta}\,d\cancel{x}$$

set $(x-\beta = y)$ $\quad = \frac{1}{2\pi}\int f(y)e^{iwy}e^{iw\beta}\,dx$

$$= e^{iw\beta}\,\mathfrak{F}[f](w)$$

Here $\mathfrak{F}^{-1}\left[e^{iw\beta}\mathfrak{F}[f](w)\right] = f(x-\beta)$

10.4.2 $u(x,t) = \int c(w)\,e^{-iwx}e^{-kw^2t}\,dw$

$$= \cancel{-\frac{e^{-iwx}}{ix}c(w)e^{-kw^2t}\Big|_{-\infty}^{\infty}} + \int \frac{e^{-iwx}}{ix}\frac{d}{dw}\left(c(w)e^{-kw^2t}\right)dw$$

Since $\lim_{w\to\pm\infty} c(w) = 0$

Now $\lim_{x\to\infty} u(x,t) = \lim_{x\to\infty}\int \frac{e^{-iwx}}{ix}\frac{d}{dw}\left(c(w)e^{-kw^2t}\right)dw \to 0$

10.4.3 (a)

$$u_t = k u_{xx} + c u_x \qquad -\infty < x < \infty$$

$$u(x,0) = f(x)$$

Let $U(w,t) = \mathfrak{F}[u(x,t)] = \frac{1}{2\pi}\int u(x,t)\, e^{iwx}\, dx$

We have: $\mathfrak{F}[u_t(\cdot,t)](w) = \frac{d}{dt} U(w,t)$

$$\mathfrak{F}[u_x(\cdot,t)](w) = (-iw)\, U(w,t)$$

$$\mathfrak{F}[u_{xx}(\cdot,t)](w) = (-iw)^2\, U(w,t) = -w^2\, U(w,t)$$

Here, in Fourier domain, PDE becomes:

$$\frac{d}{dt} U(w,t) = -kw^2\, U(w,t) - iwc\, U(w,t)$$
$$= (-kw^2 - iwc)\, U(w,t)$$

The solution of this DIFFERENTIAL EQUATION IS

$$U(w,t) = \alpha(w)\, e^{(-kw^2 - iwc)t}$$

where $\alpha(w) = U(w,0) = \mathfrak{F}[f](w) := F(w)$

Let $G_t(w) = e^{-ktw^2}$ and $g_t(x) = \mathfrak{F}^{-1}[G_t](x)$

To find $u(x,t)$, need to compute INVERSE FOURIER TRANSF. of

$$U(w,t) = F(w)\, G_t(w)\, e^{-ictw}$$

By convolution theorem, $\mathfrak{F}^{-1}[F G_t](x) = \frac{1}{2\pi} f * g_t(x)$

By Ex. 10.3.5, $\mathfrak{F}^{-1}[F G_t\, e^{-ictw}](x) = \frac{1}{2\pi} f * g_t(x+ct)$

By tables, $g_t(x) = \sqrt{\frac{\pi}{kt}}\, e^{-x^2/4kt}$

Here $u(x,t) = \frac{1}{2\pi}\int f(y)\sqrt{\frac{\pi}{kt}}\, e^{-(x+ct-y)^2/4kt}\, dy$

$$= \frac{1}{\sqrt{4\pi kt}}\int f(y)\, e^{-(x+ct-y)^2/4kt}\, dy$$