1.3.1 By Fourier's law of conduction, the heat flux is

$$\varphi(x,t) = -K_0(x)\frac{\partial u(x,t)}{\partial x}$$

If the heat flux at L is proportional to difference $u(L,t)$ and $u_B(t)$, then

$$-K_0(L)\frac{\partial u}{\partial x}(L,t) = H\left(u(x,L) - u_B(t)\right)$$

If $u(x,L) > u_B(t)$, then heat flux $\varphi(L,t)$ must be positive (net heat out flow). Thus it must be $H > 0$.

1.4.1 (a) $\dfrac{d^2u}{dx^2} = 0 \qquad u(0)=0 \qquad u(L)=T$

$u(x) = c_1x + c_2 \qquad u(0) = c_2 = 0, \quad u(L) = c_1L = T \Rightarrow c_1 = \dfrac{T}{L}$

$u(x) = \dfrac{T}{L}x$

(b) $\dfrac{d^2u}{dx^2} = 0 \qquad u(0) = T, \quad u(L) = 0$

$u(x) = c_1x + c_2 \qquad u(0) = c_2 = T, \quad u(L) = c_1L + T = 0 \Rightarrow c_1 = -\dfrac{T}{L}$

$u(x) = -\dfrac{T}{L}x + T$

(c) $\dfrac{d^2u}{dx^2} = 0 \qquad \dfrac{\partial u}{\partial x}(0) = 0 \qquad u(L) = T$

$u(x) = c_1x + c_2 \qquad u'(0) = c_1 = 0 \qquad u(L) = c_2 = T$

$u(x) = T$

(d) $\dfrac{d^2u}{dx^2} = 0 \qquad u(0) = T \qquad \dfrac{\partial u}{\partial x}(L) = \alpha$

$u(x) = c_1x + c_2 \qquad u(0) = c_2 = T \qquad \dfrac{\partial u}{\partial x}(L) = c_1 = \alpha$

$u(x) = \alpha x + T$

1.4.7 (a) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + 1 \qquad u(x,0) = f(x) \qquad \dfrac{\partial u}{\partial x}(0,t) = 1, \qquad \dfrac{\partial u}{\partial x}(L,t) = \beta$

$\dfrac{\partial u}{\partial t} = 0 \Rightarrow \dfrac{\partial^2 u}{\partial x^2} = -1 \Rightarrow u(x) = -\dfrac{x^2}{2} + c_1x + c_2$

$\dfrac{\partial u}{\partial x} = -x + c_1 \qquad \dfrac{\partial u}{\partial x}(0) = c_1 = 1 \qquad \dfrac{\partial u}{\partial x}(L) = -L + c_1 = \beta$

Need $\beta = -L + 1 \Rightarrow \beta = 1 - L$

In this case $u(x) = -\dfrac{x^2}{2} + x + c_2$

2.2.2 (a)

$$L(a_1u_1+a_2u_2)=\frac{\partial}{\partial x}\left[K_0(x)\frac{\partial}{\partial x}(a_1u_1+a_2u_2)\right]=$$

$$=\frac{\partial}{\partial x}\left[a_1K_0(x)\frac{\partial u_1}{\partial x}+a_2K_0(x)\frac{\partial u_2}{\partial x}\right]=$$

$$=a_1\frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u_1}{\partial x}\right]+a_2\frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u_2}{\partial x}\right]$$

$$=a_1L(u_1)+a_2L(u_2)$$

(b)

$$L(au)=\frac{\partial}{\partial x}\left[K_0(x,au)\frac{\partial au}{\partial x}\right]=a\frac{\partial}{\partial x}\left[K_0(x,au)\frac{\partial u}{\partial x}\right]$$

$$\neq a\frac{\partial}{\partial x}\left[K_0(x,u)\frac{\partial u}{\partial x}\right].$$

NOTE: $$\frac{\partial}{\partial x}\left[K_0(x,au)\frac{\partial u}{\partial x}\right]=\underbrace{\frac{\partial K_0}{\partial x}(x,au)}_{\neq\frac{\partial}{\partial x}K_0(x,u)}\frac{\partial u}{\partial x}+\underbrace{K_0(x,au)}_{\neq K_0(x,u)}\frac{\partial^2u}{\partial x^2}$$

2.2.4 (a) Suppose $L(u_1)=0$, $L(u_2)=0$, $L(u_p)=f$

Then: $L(u_p+c_1u_1+c_2u_2)=L(u_p)+c_1L(u_1)+c_2L(u_2)=L(u_p)=f$

(b) Let $L(u_{p_1})=f_1$, $L(u_{p_2})=f_2$

Then: $L(u_{p_1}+u_{p_2})=L(u_{p_1})+L(u_{p_2})=f_1+f_2$