HW2 SOLUTION

2.3.1 (b) $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - v_0 \frac{\partial u}{\partial x}$

$u(x,t) = \varphi(x) G(t) \Rightarrow$

$$\varphi(x)\frac{dG}{dt}(t) = k\, G(t)\frac{d^2\varphi}{dx^2} - v_0\, G(t)\frac{d\varphi}{dx}$$

$$\Rightarrow \frac{1}{kG}\frac{dG}{dt} = \frac{1}{\varphi}\frac{d^2\varphi}{dx^2} - \frac{v_0}{k}\frac{1}{\varphi}\frac{d\varphi}{dx} = -\lambda$$

$$\Rightarrow \boxed{\frac{dG}{dt} = -\lambda k G} \qquad \boxed{\frac{d^2\varphi}{dx^2} - \frac{v_0}{k}\frac{d\varphi}{dx} = -\lambda\varphi}$$

2.3.2 $\varphi''(x) + \lambda\varphi = 0$

(a) $\varphi(0) = \varphi(\pi) = 0$

As in Sec. 2.3.4, with $L = \pi$

If $\lambda > 0$, $\boxed{\varphi(x) = c\sin(nx)}$ $n = 1,2,3..$ $\boxed{\lambda = n^2}$, $n = 1,2,..$

If $\lambda \le 0$, $\varphi(x) \equiv 0$ AS IN CLASS

(b) $\varphi(0) = \varphi(1) = 0$

As in Sec. 2.3.4 with $L = 1$

If $\lambda > 0$, $\boxed{\varphi(x) = c\sin(n\pi x)}$, $n = 1,2,..$ $\boxed{\lambda = (n\pi)^2}$ $n = 0,1,..$

If $\lambda \le 0$, $\varphi(x) \equiv 0$

(c) $\varphi'(0) = \varphi'(L) = 0$

As in Sec. 2.4.1

If $\lambda > 0$, $\boxed{\varphi(x) = c\cos\left(\frac{n\pi}{L}\right)x}$ $n = 1,2..$ $\boxed{\lambda = \left(\frac{n\pi}{L}\right)^2}$, $n = 1,2...$

If $\lambda = 0$, $\varphi(x) = c$

If $\lambda < 0$, $\varphi(x) \equiv 0$

(d) $\varphi(0) = 0$, $\varphi'(L) = 0$

$\lambda > 0$ $\varphi(x) = c_1 \cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x$

$\varphi(0) = c_1 = 0$

$\varphi'(L) = \sqrt{\lambda}\, c_2 \cos\sqrt{\lambda}L = 0 \Rightarrow \sqrt{\lambda}L = \frac{2n-1}{2}\pi$

$\Rightarrow \boxed{\lambda = \left(\frac{2n-1}{2L}\pi\right)^2}$ $n = 1,2..$

$\lambda = 0$ $\varphi(x) = c_1 x + c_2$

$\varphi(0) = c_2 = 0$

$\varphi'(L) = c_1 = 0$

$\boxed{\varphi(x) = c\sin\frac{(2n-1)}{2L}\pi x}$

(e) $\varphi'(0) = \varphi(L) = 0$

$\lambda > 0$ $\varphi(x) = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x$

$\varphi'(0) = \sqrt{\lambda}\, c_2 = 0 \Rightarrow c_2 = 0$

$\varphi(L) = c_1\cos\sqrt{\lambda}L = 0 \Rightarrow$ $\sqrt{\lambda}L = \frac{2n-1}{2}\pi$

$\Rightarrow \boxed{\lambda = \left(\frac{2n-1}{2L}\pi\right)^2}$, $n = 1,2,..$

$\lambda = 0$ $\varphi(x) = c_1 x + c_2$

$\varphi'(0) = c_1 = 0$ $\varphi(L) = c_2 = 0$

$\boxed{\varphi(x) = c\cos\frac{(2n-1)}{2L}\pi x}$

2.3.3 a

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

B.C. $u(0,t) = u(L,t) = 0$

I.C. $u(x,0) = 6 \sin \frac{9\pi x}{L}$

GEN SOL. of B.C. problem is $u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$$

CLEARLY, I.C. is SATISFIED for $n = 9$, $B_9 = 6$.

Here $\boxed{u(x,t) = 6 \sin \frac{9\pi x}{L} e^{-k\left(\frac{9\pi}{L}\right)^2 t}}$

2.3.5

$$\sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} = \frac{1}{2}\left(\cos \frac{(n-m)\pi x}{L} - \cos \frac{(n+m)\pi x}{L}\right)$$

$n \neq m \Rightarrow \frac{1}{2}\int_0^L \cos \frac{(n-m)\pi x}{L} dx = \frac{1}{2} \frac{L}{(n-m)\pi} \left[\sin \frac{(n-m)\pi x}{L}\right]_0^L = \frac{1}{2} \frac{L}{(n-m)\pi} \left(\sin (n-m)\pi - 0\right) = 0$

SAME if $n \neq m$ and $(n-m)$ is replaced by $n+m$

If $n = \pm m$ then $\int_0^L \left(\sin \frac{n\pi x}{L}\right)^2 dx = \frac{1}{2}\int_0^L \left(1 - \cos \frac{2n\pi x}{L}\right) dx = \frac{L}{2} - \cancel{\frac{1}{2} \frac{L}{2n\pi} \sin \frac{2n\pi x}{L}\Big|_0^L} = \frac{L}{2}$

Here $\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \begin{cases} \frac{L}{2} & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$ (since $n, m > 0$, then it cannot be $n = -m$)

2.4.1 (d)

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

B.C. $u'(0,t) = u'(L,t) = 0$

I.C. $u(x,0) = -3 \cos \frac{8\pi x}{L}$

GEN SOL. of B.C. problem $u(x,t) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

$$u(x,0) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L}$$

I.C. is satisfied if ~~A~~ $n = 8$, $A_8 = -3$

Here $\boxed{u(x,t) = -3 \cos \frac{8\pi x}{L} e^{-k\left(\frac{8\pi}{L}\right)^2 t}}$