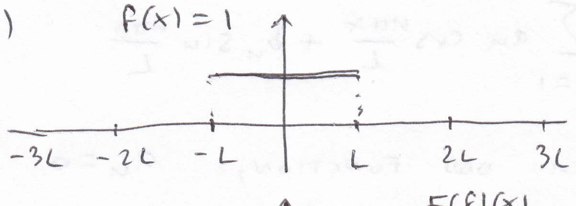
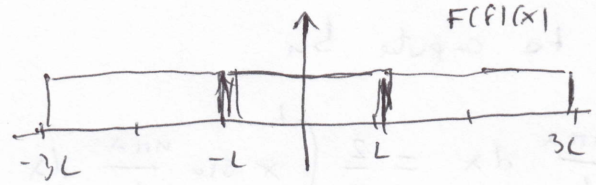
HW #3 SOLUTION

3.2.1

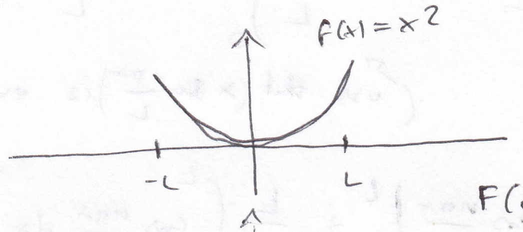
(a)

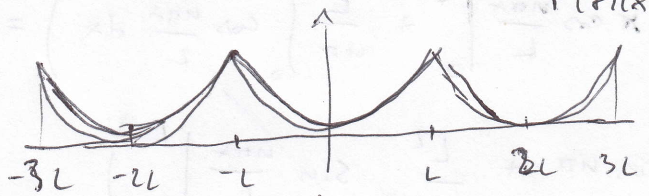


$f(x) = F(f)(x) \quad \forall x \in [-L,L]$



(b)

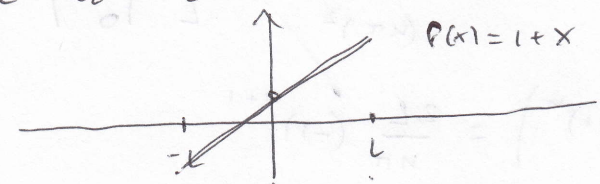


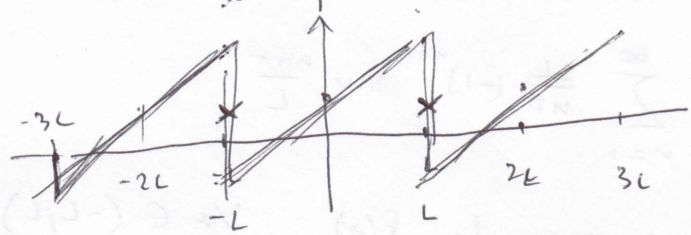


$f(x) = F(f)(x) \quad \forall x \in [-L,L]$

NOTE: $f(-L) = f(L)$

(c)





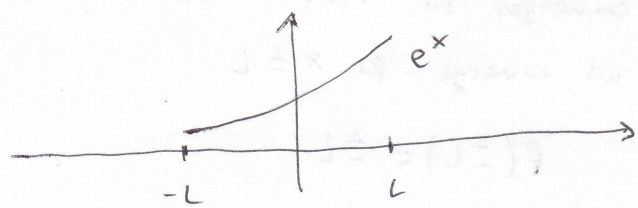
$f(x) = F(f)(x) \quad \forall x \in (-L,L)$

$f(x) \neq F(f)(x)$ at $x = \pm L$

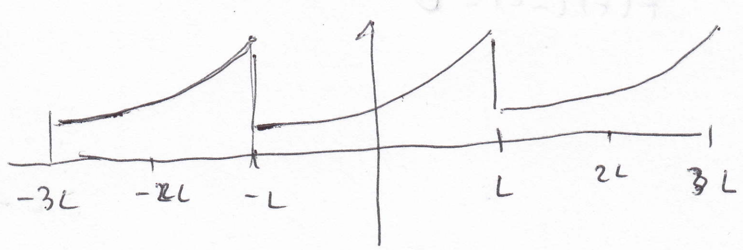
$f(\pm L) = 1 \pm L$

$F(f)(\pm L) = 1$

(d)



$f(x) = F(f)(x) \quad \forall x \in (-L,L)$



$f(x) \neq F(f)(x)$ at $x \pm L$

$f(\pm L) = e^{\pm L}$

$F(f)(\pm L) = \frac{1}{2}(e^{L} + e^{-L})$

3.2.2 (a) $\quad f(x) = x \quad -L \le x \le L$

$$F(f)(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

Since x is an ODD FUNCTION, $a_n = 0$ for all $n \ge 0$

We only need to compute b_n

$$b_n = \frac{1}{L}\int_{-L}^{L} x \sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L x \sin\frac{n\pi x}{L}\,dx =$$

(use that $\left(x \sin\frac{n\pi x}{L}\right)$ is even)

$$= \frac{2}{L}\left(-\frac{L}{n\pi} x \cos\frac{n\pi x}{L}\Big|_0^L + \frac{L}{n\pi}\int_0^L \cos\frac{n\pi x}{L}\,dx\right) =$$

$$= \frac{2}{L}\left(-\frac{L^2}{n\pi}\cos n\pi + \frac{L^2}{(n\pi)^2}\sin\frac{n\pi x}{L}\Big|_0^L\right)$$

$$= \frac{2}{L}\left(-\frac{L^2}{n\pi}(-1)^n\right) = \frac{2L}{n\pi}(-1)^{n+1}$$

Hence $\displaystyle F(f)(x) = \sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^{n+1} \sin\frac{n\pi x}{L}$

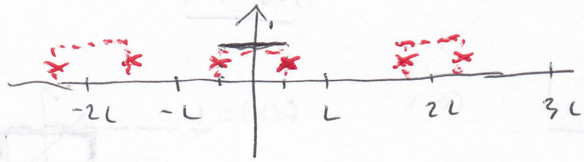
NOTE: $F(f)(x)$ converges to $f(x)$ $\forall x \in (-L, L)$.

It does not converge for $x \pm L$

$$f(\pm L) = \pm L$$

$$F(f)(\pm L) = 0$$

3.2.2 (e) $\quad f(x) = \begin{cases} 1 & |x| < L/2 \\ 0 & |x| \geq L/2 \end{cases}$



Since $f(x)$ is even, $b_n = 0$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^{L/2} dx = \frac{1}{2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^{L/2}\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\,\frac{L}{n\pi}\left(\sin\frac{n\pi x}{L}\right)_0^{L/2} =$$

$$= \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

if n even $\quad n = 2m \qquad a_{2m} = \frac{1}{m\pi}\sin m\pi = 0 \qquad \forall m$

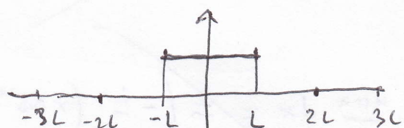
if n odd $\quad n = 2m-1 \qquad a_{2m-1} = \frac{2}{(2m-1)\pi}\sin\frac{(2m-1)\pi}{2} = \frac{2}{(2m-1)\pi}(-1)^{m+1}$

thus $\quad \boxed{F(f)(x) = \frac{1}{2} + \sum_{m=1}^{\infty}\frac{2}{(2m-1)\pi}(-1)^{m+1}\cos\frac{(2m-1)\pi x}{L}}$

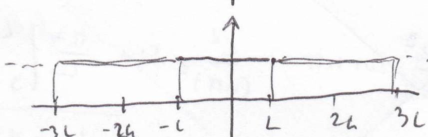
$F(f)(x) = f(x) \quad \forall x \in [-L, L]$ except $x = \pm\frac{L}{2}$

$f(\pm\frac{L}{2}) = 0 \qquad F(f)(\pm\frac{L}{2}) = \frac{1}{2}$

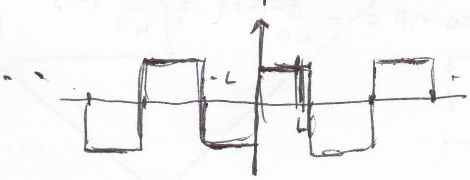
3.3.1 (a)



$f(x)$

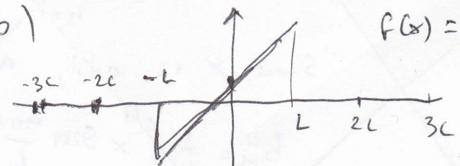


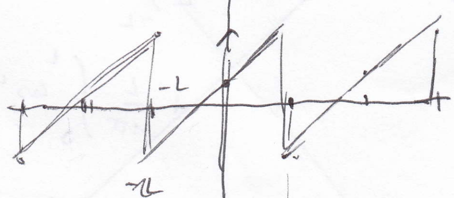
$F(f)(x)$ $\parallel$ cosine series of f



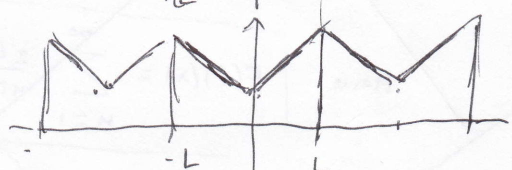
Sine series of f

(b) $\quad f(x) = 1 + x$

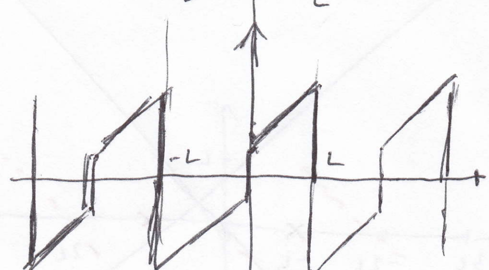




$F(f)(x)$



Cosine series of f



Sine series of f

3.3.2 (a) $f(x) = \cos\frac{x\pi}{L} \quad 0 \le x \le L$

Sine series of f = Fourier series of odd extension of f

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

where $b_n = \frac{2}{L}\int_0^L \cos\frac{\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_0^L \left(\sin\frac{(n+1)\pi x}{L} + \sin\frac{(n-1)\pi x}{L}\right)dx =$

(RECALL: $\sin A\cos B = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$)

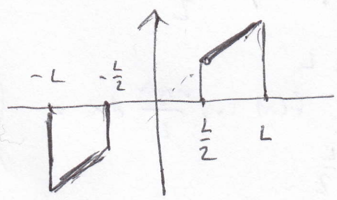
$$= \frac{1}{L}\left(\frac{L}{(n+1)\pi}\left(-\cos\frac{(n+1)\pi x}{L}\right)\Big|_0^L + \frac{L}{(n-1)\pi}\left(-\cos\frac{(n-1)\pi x}{L}\right)\Big|_0^L\right)$$

$$= \frac{1}{(n+1)\pi}\left(1-\cos(n+1)\pi\right) + \left(1-\cos(n-1)\pi\right)\frac{1}{(n-1)\pi}$$

If n = odd, then $\cos(n+1)\pi = \cos(n-1)\pi = 1 \Rightarrow b_n = 0$

If n = even, then $\cos(n+1)\pi = \cos(n-1)\pi = -1 \Rightarrow b_n = \frac{2}{(n+1)\pi} + \frac{2}{(n-1)\pi} = \frac{4n}{\pi(n^2-1)}$

3.3.2 (c) $f(x) = \begin{cases} 0, & x < \frac{L}{2} \\ x, & x > \frac{L}{2} \end{cases}$



Sine series: $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$

when $b_n = \frac{2}{L}\int_{\frac{L}{2}}^{L} x\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\left(\frac{L}{n\pi}x\left(-\cos\frac{n\pi x}{L}\right)\Big|_{\frac{L}{2}}^{L} + \frac{L}{n\pi}\int_{\frac{L}{2}}^{L}\cos\frac{n\pi x}{L}\,dx\right)$

$$= \frac{2}{L}\left(\frac{L}{n\pi}\frac{L}{2}\cos\frac{n\pi}{2} - \frac{L\cdot L}{2n\pi}\cos n\pi + \frac{L}{n\pi}\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_{\frac{L}{2}}^{L}\right)$$

$$= \frac{2}{L}\left(\frac{L^2}{2n\pi}\cos\frac{n\pi}{2} - \frac{L^2}{2n\pi}\cos n\pi + \cancel{\frac{L^2}{(n\pi)^2}\sin n\pi} - \frac{L^2}{(n\pi)^2}\sin\frac{n\pi}{2}\right)$$

$$= \frac{L}{n\pi}\cos\frac{n\pi}{2} - \frac{L}{n\pi}\cos n\pi - \frac{L}{2(n\pi)^2}\sin\frac{n\pi}{2}$$

If n = odd, then $b_n = \frac{L}{n\pi} - \frac{L}{2(n\pi)^2}\sin\frac{n\pi}{2}$

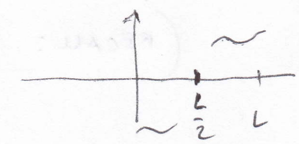
If n = even, then $b_n = \frac{L}{n\pi}\cos\frac{n\pi}{2} - \frac{L}{n\pi}$

3.3.7

$$f(x) = \underbrace{\frac{1}{2}\left(f(x) + f(-x)\right)}_{\uparrow \text{ EVEN}} + \underbrace{\frac{1}{2}\left(f(x) - f(-x)\right)}_{\uparrow \text{ ODD}}$$

$$e^x = \underbrace{\frac{1}{2}\left(e^x + e^{-x}\right)}_{\text{even}} + \underbrace{\frac{1}{2}\left(e^x - e^{-x}\right)}_{\text{odd}} = \cosh x + \sinh x$$

3.3.15 Suppose f is ODD around $x = L/2$



This implies that $\int_0^L f(x) = 0 \Rightarrow a_0 = 0$

$$n \geq 1: \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\left(\int_0^{\frac{L}{2}} f(x)\cos\frac{n\pi x}{L}\,dx + \int_{\frac{L}{2}}^{L} f(x)\cos\frac{n\pi x}{L}\,dx\right)$$

Set $x = y + \frac{L}{2}$ in second integral

then
$$\int_{\frac{L}{2}}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \int_0^{\frac{L}{2}} f\left(y+\frac{L}{2}\right)\cos\frac{n\pi\left(y+\frac{L}{2}\right)}{L}\,dy =$$

$$= \int_0^{\frac{L}{2}} f\left(y+\frac{L}{2}\right)\cos\left(\frac{n\pi y}{L}+\frac{n\pi}{2}\right)dy$$

NOTE: $f\left(y+\frac{L}{2}\right) = -f(y)$
since f odd
around $y = \frac{L}{2}$

$$= -\int_0^{\frac{L}{2}} f(y)\cos\left(\frac{n\pi y}{L}+\frac{n\pi}{2}\right)dy$$

Hence:
$$a_n = \frac{2}{L}\left(\int_0^{L} f(x)\cos\frac{n\pi x}{L}\,dx - \int_0^{L} f(y)\cos\left(\frac{n\pi y}{L}+\frac{n\pi}{2}\right)dy\right)$$

If $n = 4m$, then $\cos\left(\frac{n\pi y}{L}+\frac{n\pi}{2}\right) = \cos\left(\frac{4m\pi y}{L}+2m\pi\right) = \cos\left(\frac{4m\pi y}{L}\right)$

Hence $a_n = 0$